

# QCD effects in the $D^o\overline{D}_o^*$ component of $X(3872)$

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## Abstract

The dynamical effects of the QCD-based gluon field overlap factor  $f$  ( $\exp(-b_s k_f \sum_{i<j} r_{ij}^2)$ ) has been seen for the scattering process of  $D^o\overline{D}_o^* \rightarrow \omega J/\psi$ . The value used for the parameter  $k_f$  in the exponent of the factor- $f$  has its origin in numerical lattice simulations. We have calculated the cross sections and phase shifts for the scattering process of  $D^o\overline{D}_o^* \rightarrow \omega J/\psi$  when  $f$  is used in the off-diagonal elements of potential energy, kinetic energy and normalization matrices of a model of  $q^2\overline{q}^2$  systems in the gluonic cluster basis. Our cross sections can be potentially compared with experiments. As  $D^o\overline{D}_o^*$  is considered to be a significant component of the meson  $X(3872)$ , our work may have implications for the structure of  $X(3872)$  in pointing out the roles of this component of it and other in it; we accordingly comment on meson-meson binding in the  $D^o\overline{D}_o^*$ . By using the resonating group method, we hope to explore the features of the meson-meson system possibly missed in Born-approximation based treatments for the almost degenerate states of  $D^o\overline{D}_o^*$  and  $X(3872)$ .

## 1 Introduction

In the last few years important progress has been made as far as charmonium spectroscopy is concerned. The many missing states, which were expected in quark model, has been seen. Furthermore some unexpected states has also been discovered experimentally which are challenging the quark model seriously. These newly born states were given names alphabetically as XYZ [1]. An important member of this family is the meson  $X(3872)$  which is now generally considered [2] as a mixture of  $D^o\overline{D}_o^*$ ,  $D^+\overline{D}_-^*$  and  $c\overline{c}$ . Any effort to understand it, thus, should understand many quantities composed of combinations of its components. A direct lattice study of it would have to calculate many corresponding Wilson loops before arriving at any conclusion. A more manageable route could be to make separate models of its components, find out their consequences and then combine the models to understand  $X(3872)$  or other mesons with a complex structure like that. If separate models use Hamiltonians, then for combining the components of  $X(3872)$  the suitable choice perhaps remains a non-relativistic quantum-mechanics-based formalism.

Here we take the first step in this scheme and concentrate on models of  $D^o\overline{D}_o^*$ . This is a two quarks two antiquarks molecule. We have already worked on meson-meson molecules [3]. But that work did not take into account the spin and flavour degrees of freedoms. We now specialize the formalism developed to the charm, anticharm quarks along with lighter ones. That is what we have to do study the  $D^o\overline{D}_o^*$  system. (Perhaps it is worth mentioning here that there are other meson molecules worth studying as well. Rather the strong candidates for meson molecules or tetraquark states belong to light scalars [4] [5] [6] as well as some hidden charm resonances [7] e.g.  $X(3872)$ ,  $Y(4260)$ ,  $Y(4140)$  and  $Z^+(4430)$ . Some times a common name 'tetraquark' is used for both the types ( $q\overline{q}q\overline{q}$  and  $\overline{q}\overline{q}qq$ ) because the lattice methods do not distinguish between the two types [8].

"Scalar mesons are special because they have the same quantum numbers as the quantum numbers of vacuum,  $J^{PC} = 0^{++}$  [9]. So in contrast to pseudo-scalar and vector mesons, the scalar mesons have non zero vacuum expectation values. They belong to Higgs sector of strong interactions and reflect the QCD-vacuum structure." "By taking scalar mesons as  $q\overline{q}$ -systems in the naive constituent quark models, it is not possible to reproduce the mass spectrum of the scalar nonet [10]. (There have been observed

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15 scalar mesons including  $f_0(400-1200)$ ,  $f_0(1370)$ ,  $f_0(1500)$ ,  $f_0(1710)$ ,  $k_0^*(1430)$ ,  $a_0(980)$ ,  $a_0(1430)$  [11] instead of 9 below 2  $GeV$ .) Thus whereas SU(3) flavour symmetry works well for reproducing the mass spectrum of pseudoscalar ( $J^{PC} = 0^{-+}$ ) and vector ( $J^{PC} = 1^{--}$ ) meson nonets in the conventional  $q\bar{q}$  picture of mesons, this  $q\bar{q}$  picture is not working for scalar mesons [12]. The excess of scalar mesons may be explained by the introduction of non-conventional mesons with internal structure other than the  $q\bar{q}$ . So the possible interpretation [10] of the internal structure may be: 1- "They may be four quark states with strong diquark correlations" [4] [13] [14] [15] 2- "They may be collective  $q\bar{q}$  excitations" [16] [17] [18] [19] [20] [21] 3- "They may be mesonic molecules generated dynamically" [22] [23] [24] [25] [26] [27] 4- "They may be glueballs" [28].)

Coming back to  $X(3872)$ , there are indications that  $X(3872)$ ,  $Y(4140)$  and  $Y(4260)$  are tetra-quark or molecular states [8]. Author of ref. [29] identified  $X(3872)$  being a tetraquark. Ref. [30] assume that  $X(3872)$  is a molecular bound state of neutral charm mesons. Not everyone, though, takes the  $X$  to be purely molecule or tetraquark. But generally it is accepted [2][31] that the  $D^0\bar{D}_0^*$  molecule is a significant (if not dominant) component of  $X(3872)$ . (We would like to add that such of mixing of a quark antiquark state with a meson-meson molecule is not very uncommon: In ref. [32] it is stated that the "Fock space decomposition of the  $\sigma$  meson reveals that the dominant component behaves with the increase of  $N_c$ , up to  $N_c = 6$ , as a  $q\bar{q}q\bar{q}$  then the subdominant  $q\bar{q}$ -like takes over for large  $N_c$  up to  $N_c = 20$ . The glueball-like component stays always at or below the 10% level". The glue ball states do not mix with  $q\bar{q}$ ,  $q\bar{q}q\bar{q}$  etc. in quenched approximation [33]. However full QCD reveals that there is mixing between  $u\bar{u} + d\bar{d}$  states and  $s\bar{s}$  state along with glueball and multi-quark channels for any value of  $J^{PC}$  of flavour singlet states. So it turns out that for the physical spectrum of scalar mesons there is a mixing of  $q\bar{q}$ ,  $q\bar{q}q\bar{q}$ , glueball etc. [34].)

For understanding  $X(3872)$  we have selected to discuss the  $D^0\bar{D}_0^*$  molecule. This is a pseudoscalar vector structure, which is also important in its own because a)  $P_s V$  (pseudoscalar vector) scattering gives excellent information about the spin dependent forces b)  $P_s V$  scattering phase shifts are experimentally accessible [35]. In  $P_s V$  scattering a  $D^0\bar{D}_0^{o*}$  system may go to an  $\omega J/\psi$  system by making an intermediate state  $X(3872)$ .

There are many works [27][36][37][38][31] that treat two-quarks two antiquarks through a sum of pair-wise interaction and calculate the meson-meson binding in it. For example, Swanson found [27] that a  $D\bar{D}^*$  system may couple to  $\omega J/\psi$  system in the one gluon exchange model but, according to him, the small interaction is not sufficient for a bound molecular state. Through the present work, we want to find implications of non only improving the two-quarks two antiquark modeling beyond the sum of two body approach but work out a formalism for meson meson scattering that is not limited to Born approximation. In a recent work of us [3], we have mentioned a model that is better than a simple sum of two body approach and have worked out its meson-level consequences. We had to use a Born approximation there. In the present paper, we use a resonating group formalism to go beyond the Born approximation and introduce the spin and flavour degrees of freedoms. A similar realistic meson-meson treatment for lighter quarks has been published earlier [39]. But, in contrast to [39] we now address a meson-meson system ( $D^0\bar{D}_0^*$ ) of current interest. We also give, or plan to give, a much more thorough analysis of the meson-meson phase shifts and binding in this system. We include also the cross-section that are not in [39] at all.

As compared to the sum-of-two body approach, we show in our results section how the inclusion of the QCD effect in the form of the gluonic field overlap factor  $f$  reduces both the phase shifts and cross sections, meaning a reduced interaction.

We should point out in one respect our present two quarks two antiquarks model is not as close to QCD as is the model used in [3]. This is because, for the gluonic field factor  $f$  of the model, we use an old [40] but computationally more convenient Gaussian form. It is expected to be a good approximation because we have checked that the binding energies for a four quark system do not change much if we use the Gaussian form of  $f$  or more refined surface area form [41] of  $f$ .

We think our improvement beyond the Born approximation is an important one. This is because we are dealing with states ( $D^0\bar{D}_0^*$ ),  $J/\psi\omega$  and  $X(3872)$  that are close in energy. Energy coincidence is the main reason behind a mixing through quark exchange in  $D^0\bar{D}_0^{o*} \rightarrow J/\psi u\bar{u}$  because  $D^0\bar{D}_0^{o*}$  and  $J/\psi\omega$  are effectively mass degenerate. For such almost degenerate states approaches similar to a non-degenerate perturbation theory (or Born approximation) fail; see ref. [42]. As our resonating group method based treatment is not limited to any perturbation theory (or it has some resemblance with a degenerate perturbation theory) we hope to include effects that might have been missed in treatments

(like Born approximation) that that closer to a non-degenerate perturbation theory.

In section 2 we have defined the  $Q^2\bar{Q}^2$  Hamiltonian and the corresponding wave function with reference to spin and flavor basis. There we have also given the form of  $f$  that we have used. In sec. 3 we have calculated the Potential Energy, Kinetic Energy and Overlap Matrices in the Flavor, Spin and Gluonic bases. In sec. 4 we have written the coupled integral equations for  $Q^2\bar{Q}^2$  system and solved them for transition amplitudes. In sec. 5 we have presented our results and gave conclusions.

## 2 The $Q^2\bar{Q}^2$ Hamiltonian and the Wave-function

In ref. [27] it is given that the Born order scattering amplitude for  $D^o\bar{D}^{o*} \rightarrow \omega J/\psi$  scattering is dominated by the confinement interaction (in the present work we are taking quadratic confinement) which is in contrast to light meson scattering (which is dominated by the hyperfine interaction) so hyperfine interaction for this process can be neglected which we did here in the present work.

Using adiabatic approximation we can write the total state vector of a system containing two quarks two antiquarks and the gluonic field between them as a sum of product of quarks ( $Q$  or  $\bar{Q}$ ) position dependence function  $\Psi_r(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$  and the gluonic field state  $|k\rangle_g$  along with flavor and spin bases.  $|k\rangle_g$  is defined as a state which approaches  $|k\rangle_c$  in the weak coupling limit, with  $|1\rangle_c = |1_{1\bar{3}}1_{2\bar{4}}\rangle_c$ ,  $|2\rangle_c = |1_{1\bar{4}}1_{2\bar{3}}\rangle_c$  and  $|3\rangle_c = |\bar{3}_{12}3_{\bar{3}\bar{4}}\rangle_c$ . In lattice simulations of the corresponding (gluonic) Wilson loops it is found that the lowest eigenvalue of the Wilson matrix, that is energy of the lowest state, is always the same for both  $2 \times 2$  and  $3 \times 3$  matrices provided that  $|1\rangle_g$  or  $|2\rangle_g$  has the lowest energy [41]. The later calculations [43] of the tetraquark system were also done with a two level approximation. Taking advantage of these observations, we have included in our expansion only two basis states. As in resonating group method,  $\Psi_r(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$  or  $\Psi_r(\mathbf{R}_c, \mathbf{R}_k, \mathbf{y}_k, \mathbf{z}_k)$  is written as product of known dependence on  $\mathbf{R}_c, \mathbf{y}_k, \mathbf{z}_k$  and unknown dependence on  $\mathbf{R}_k$ . i.e.  $\Psi_r(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \Psi_c(\mathbf{R}_c)\chi_k(\mathbf{R}_k)\psi_k(\mathbf{y}_k, \mathbf{z}_k)$ . Here  $\mathbf{R}_c$  is the center of mass coordinate of the whole system,  $\mathbf{R}_1$  is the vector joining the center of mass of the clusters  $(1, \bar{3})$  and  $(2, \bar{4})$ ,  $\mathbf{y}_1$  is the position vector of quark 1 with respect to  $\bar{3}$  within the cluster  $(1, \bar{3})$  and  $\mathbf{z}_1$  is the position vector of quark 2 with respect to  $\bar{4}$  within the cluster  $(2, \bar{4})$ . The same applies to  $\mathbf{R}_2, \mathbf{y}_2$  and  $\mathbf{z}_2$  for the clusters  $(1, \bar{4})$  and  $(2, \bar{3})$ . Similarly we can define  $\mathbf{R}_3, \mathbf{y}_3$  and  $\mathbf{z}_3$  for the clusters  $(1, 2)$  and  $(\bar{3}, \bar{4})$ . Or we can write them in terms of position vector of the four particles (quarks or antiquarks) as follow.

$$\mathbf{R}_1 = \frac{(\mathbf{r}_1 + r\mathbf{r}_3 - r\mathbf{r}_2 - \mathbf{r}_4)}{(1+r)} \quad (1)$$

$$\mathbf{R}_3 = \frac{(\mathbf{r}_1 + r\mathbf{r}_2 - r\mathbf{r}_3 - \mathbf{r}_4)}{(1+r)} \quad (2)$$

Where  $r = \frac{m_c}{m}$ ,  $m_c$  is constituent mass of *charm* quark and  $m$  is that of *up* or *down* quark. While the other definitions of position vectors are

$$\mathbf{y}_1 = \mathbf{r}_1 - \mathbf{r}_3 \text{ and } \mathbf{z}_1 = \mathbf{r}_2 - \mathbf{r}_4, \quad (3)$$

$$\mathbf{R}_2 = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_4 - \mathbf{r}_2 - \mathbf{r}_3), \mathbf{y}_2 = \mathbf{r}_1 - \mathbf{r}_4 \text{ and } \mathbf{z}_2 = \mathbf{r}_2 - \mathbf{r}_3 \quad (4)$$

and

$$\mathbf{y}_3 = \mathbf{r}_1 - \mathbf{r}_2 \text{ and } \mathbf{z}_3 = \mathbf{r}_3 - \mathbf{r}_4. \quad (5)$$

Thus meson meson state vector in the restricted gluonic basis is written as

$$|\Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4; g)\rangle = \sum_{k=1}^2 |k\rangle_g |k\rangle_f |k\rangle_s \Psi_c(\mathbf{R}_c) \chi_k(\mathbf{R}_k) \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k). \quad (6)$$

The interquark potential in each cluster is taken as simple harmonic oscillator potential (see eq. (12)). So the space part of mesonic wave functions remain Gaussian i.e.

$$\xi_k(\mathbf{y}_k) = \frac{1}{(2\pi d_{k1}^2)^{\frac{3}{4}}} \exp\left(\frac{-\mathbf{y}_k^2}{4d_{k1}^2}\right) \quad (7)$$

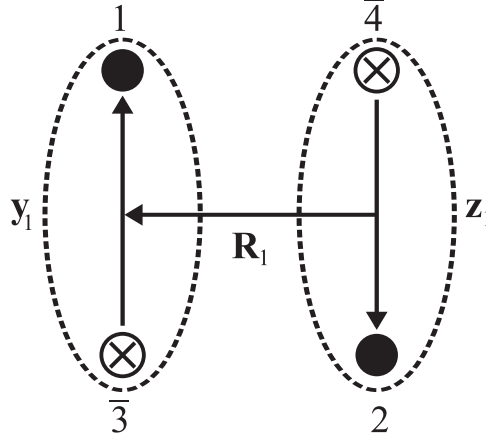


Figure 1: Topology1.

and

$$\zeta_k(\mathbf{z}_k) = \frac{1}{(2\pi d_{k2}^2)^{\frac{3}{4}}} \exp\left(\frac{-\mathbf{z}_k^2}{4d_{k2}^2}\right), \quad (8)$$

but with different sizes.  $k=1,2$  correspond to topologies in fig. 1 and fig. 2 respectively, while  $d_{k1}$  and  $d_{k2}$  are the sizes of mesonic wave functions. For a 3-d simple harmonic oscillator, size square is inversely proportional to the square root of reduced mass of quark and antiquark in the meson. So, if

$d = \left(\frac{1}{\frac{m}{m} + \frac{m}{m}}\right)^{\frac{1}{4}}$ ,  $d' = \left(\frac{1}{\frac{m}{m_c} + \frac{m}{m_c}}\right)^{\frac{1}{4}}$  and  $d'' = \left(\frac{1}{\frac{m_c}{m_c} + \frac{m_c}{m_c}}\right)^{\frac{1}{4}}$  (we are taking constant of proportionality being 1, this constant of proportionality eliminates in the following ratios of  $d'^s$ ), then we have

$$\frac{d'^2}{d^2} = \sqrt{\frac{m(m_c + m)}{2mm_c}} = \sqrt{\frac{r+1}{2r}} \quad (9)$$

and

$$\frac{d''^2}{d^2} = \sqrt{\frac{2m}{2m_c}} = \sqrt{\frac{1}{r}} \quad (10)$$

Here  $d' = d_{11} = d_{12}$  is size of mesonic wave function of a heavy-light quark system corresponding to mesons ( $D^o \bar{D}^{o*}$ ) in channel-1,  $d = d_{21}$  for both light quarks and  $d'' = d_{22}$  is correspondingly for both heavy quarks for mesons ( $\omega$  and  $J/\psi$  respectively) in channel-2.

These Gaussian forms of meson wave functions are, strictly speaking, the wave functions of a quadratic confining potential. But, as pointed out in text below Fig. 1 of ref. [44], the overlap of a Gaussian wave function and the eigenfunction of the realistic linear plus colombic potential can be made as close as 99.4% by properly adjusting its parameter  $d$ . A realistic value of  $d$  mimicking a realistic meson wave function depends on the chosen scattering mesons and thus in this present work we actually have chosen the realistic value of  $d$  for  $J/\Psi$ .

As for the Hamiltonian, for  $f=1$  the total Hamiltonian  $H$  of our 4-particle system is taken as [26]

$$\hat{H} = \sum_{i=1}^4 \left[ m_i + \frac{\hat{p}_i^2}{2m_i} \right] + \sum_{i < j} v(\mathbf{r}_{ij}) \mathbf{F}_i \cdot \mathbf{F}_j. \quad (11)$$

Where  $m_i$  ( $i = 1, 2, \bar{3}, \bar{4}$ ) is the mass of  $i$ th quark and our kinetic energy operator is non-relativistic. In above  $\mathbf{F}$  has 8 components  $F_{i'} = \frac{\lambda_{i'}}{2}$ ,  $i' = 1, 2, 3, \dots, 8$  and  $F_{i'}^* = \frac{\lambda_{i'}^*}{2}$ ,  $\lambda_{i'}$  are Gell-Mann matrices.

For the pairwise  $q\bar{q}$  potential, we have used a quadratic confinement

$$v_{ij} = Cr_{ij}^2 + \bar{C} \text{ with } i, j = 1, 2, \bar{3}, \bar{4}. \quad (12)$$

As for the within-a-cluster dependence of the wave function, this use of the quadratic potential in place of the realistic Coulumbic plus linear may change the full wave function. In the within-a-cluster, this

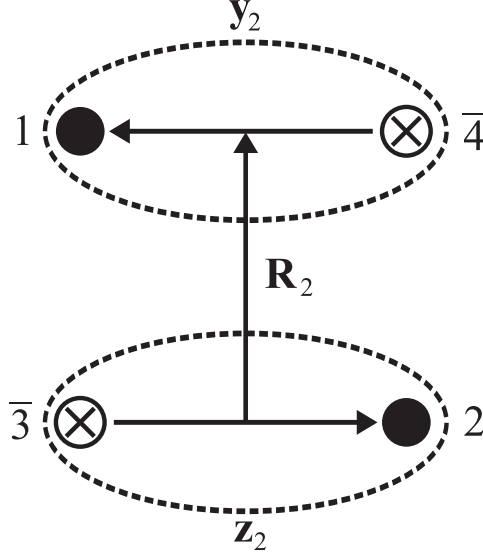


Figure 2: Topology2

change of wave function is found to result in a change of an overlap integral from 100% to 99.4% only provided the parameter  $d$  of the wave function is adjusted. It seems that by a proper adjustment the parameters of the quadratic (or SHO) model can reasonably simulate a  $q\bar{q}$  or even a  $q^2\bar{q}^2$  system. In our case, this adjustment of the parameters has been made by a choice of actual scattering mesons in the presence of spin and flavour degrees of freedom.

## 2.1 Inclusion of Spin and Flavor

In this chapter we are considering a realistic system  $D^o\bar{D}^{o*}$  by introducing spin and flavour in our formalism of treating a four quark system. But for computational convenience we are considering the gaussian form of  $f$  (see eq. (24)) instead of more accurate minimal area form (see eq. (23)). A  $q^2\bar{q}^2$  system can have a spin of 0,1 or 2. Here we study  $D^o$  and  $\bar{D}^{o*}$  with spin 0 and 1 respectively. So the total spin in the incoming channel is 1. But for the outgoing channel we have  $\omega$  and  $J/\psi$  both with spin 1. The conservation of spin tells that the spin of the outgoing channel must be 1. Our spin states are

$$|1\rangle_s = |P_{1\bar{3}}V_{2\bar{4}}\rangle \quad (13)$$

and

$$|2\rangle_s = |V_{1\bar{4}}V_{2\bar{3}}\rangle, \quad (14)$$

where  $P$  represents a pseudo-scalar and  $V$  represents a vector meson. For the flavour basis we have  $D^o = |c\bar{u}\rangle$ ,  $\bar{D}^{o*} = |\bar{c}u\rangle$ ,  $\omega = \frac{1}{\sqrt{2}}|u\bar{u} + d\bar{d}\rangle$  and  $J/\psi = |c\bar{c}\rangle$  so the flavour states in channel 1 and 2 are

$$|1\rangle_f = |c\bar{u}\rangle|\bar{c}u\rangle \quad (15)$$

and

$$|2\rangle_f = \frac{1}{\sqrt{2}}|u\bar{u} + d\bar{d}\rangle|c\bar{c}\rangle \quad (16)$$

respectively.

Now the spin flavour free resonating group method based integral equations generalize to

$$\sum_{l=1}^2 \int d^3\mathbf{y}_k d^3\mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k)_f \langle k|_s \langle k|_g \langle k| (H - E_c) | l\rangle_g | l\rangle_s | l\rangle_f \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) = 0, \quad (17)$$

for  $k = 1, 2$ . Here  $H = K + V^p + \sum_{i=1}^4 m_i$ . The gluonic states are defined in sec. 2 while spin and flavour states are defined in eqs. (13-16) above. As operator  $(H - E_c)$  is identity in the flavor basis, the overlap

factors are also the flavor matrix elements of the  $(H - E_c)$  operator. Thus the above eq. can be written as

$$\sum_{l=1}^2 \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k f \langle k | l \rangle_f \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k)_s \langle k | g \rangle_k \langle k | H - E_c | l \rangle_g | l \rangle_s \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) = 0 \quad (18)$$

for  $k = 1, 2$ .

Or

Because of isospin conservation and no annihilation effects each flavour overlap factor is unity. We may separate flavour overlap from spin and gluonic overlaps.

$$\sum_{l=1}^2 \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k)_s \langle k | g \rangle_k \langle k | H - E_c | l \rangle_g | l \rangle_s \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) = 0 \quad (19)$$

According to the (2 dimensional basis) model  $I_a$  of ref. [41], the normalization, potential energy and kinetic energy matrices in the corresponding gluonic basis only are

$$N = \begin{pmatrix} 1 & \frac{1}{3}f \\ \frac{1}{3}f & 1 \end{pmatrix}, \quad (20)$$

$$V = \begin{pmatrix} \frac{-4}{3}(v_{1\bar{3}} + v_{2\bar{4}}) & \frac{4}{9}f(v_{12} + v_{\bar{3}\bar{4}} - v_{1\bar{3}} - v_{2\bar{4}} - v_{1\bar{4}} - v_{2\bar{3}}) \\ \frac{4}{9}f(v_{12} + v_{\bar{3}\bar{4}} - v_{1\bar{3}} - v_{2\bar{4}} - v_{1\bar{4}} - v_{2\bar{3}}) & \frac{-4}{3}(v_{1\bar{4}} + v_{2\bar{3}}) \end{pmatrix} \quad (21)$$

and

$${}_g \langle k | K | l \rangle_g = N(f)^{\frac{1}{2}}_{k,l} \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m} \right) N(f)^{\frac{1}{2}}_{k,l}. \quad (22)$$

This is the modification, through the  $f$  factor, to the Hamiltonian.

## 2.2 Form of $f$

Ref. [41] supports through a comparison with numerical lattice simulations a form  $f$  that was earlier [45] suggested through a quark-string model extracted from the strong coupling lattice Hamiltonian gauge theory. This is

$$f = \exp(-b_s k_f S), \quad (23)$$

$S$  being the area of minimal surface bounded by external lines joining the position of the two quarks and two antiquarks, and  $b_s = 0.18 \text{ GeV}^2$  is the standard string tension [46, 47],  $k_f$  is a dimensionless parameter whose value of 0.57 was decided in ref. [41] by a fit of the simplest two-state area-based model (termed model Ia) to the numerical results for a selection of  $Q^2 \bar{Q}^2$  geometries. It is shown there [41] that the parameters, including  $k_f$ , extracted at this  $SU(2)_c$  lattice simulation with  $\beta = 2.4$  can be used directly in, for example, a resonating group calculation of a four quark model as the continuum limit is achieved for this value of  $\beta$ .

For interpreting the results in terms of the potential for the corresponding single heavy-light meson  $(Q\bar{q})$ , a Gaussian form

$$f = \exp(-b_s k_f \sum_{i < j} r_{ij}^2) \quad (24)$$

of the gluonic field overlap factor  $f$  is used in ref. [48] for numerical convenience and not the minimal area form. But for a particular geometry, the two exponents (the minimal area and the sum of squared distances) in these two forms of  $f$  are related and thus for a particular geometry a comparison of the parameter  $k_f$  multiplying area and corresponding (different!)  $k_f$  multiplying sum of squares in eq. (14) of ref. [48] is possible. We note that, after correcting for a ratio of 8 between the sum of distance squares (including two diagonals) and the area for the square geometry, the colour-number-generated relative difference for this geometry is just 5%: the coefficient is  $0.075 \times 8 = 0.6$  multiplying sum of squared distances and 0.57 multiplying the minimal area. But, as the precise form of  $f$  is still under development (the latest work [43] has covered only a very limited selection of the positions of tetraquark constituents) and the expression for the area in its exponent needs improvement, it is not sure precisely what value of

the  $k_f$  best simulates QCD. Furthermore we have checked that the binding energies of four-quark system do not vary much at least for the square geometry. So for the present work we are taking  $f$  reported in eq. (14) of ref. [48] and is numerically easier to deal with. This sec. 2.2 has been taken from our previous work [3].

### 3 Potential Energy, Kinetic Energy and Overlap Matrices in the flavor, spin and gluonic basis

We define in this section the potential energy, kinetic energy and the overlap matrices of potential energy, kinetic energy and identity operators in the flavor spin and gluonic basis with out doing any space differentiations and integrations. First we discuss the matrix elements of potential operator

$$\hat{V}^P(q_1 q_2 \bar{q}_3 \bar{q}_4) = \sum_{i < j}^4 \mathbf{F}_i \cdot \mathbf{F}_j v_{ij} \quad (25)$$

, where  $v_{ij}$  is defined in eq. (12), in the gluonic-spin bases.

$$V_{k_{gs}, l_{gs}}^P = {}_{gs} \langle k | \hat{V}^P | l \rangle_{gs} = \sum_{i < j} \mathbf{F}_i \cdot \mathbf{F}_j (v_{ij})_{k_{gs}, l_{gs}} = \sum_{i < j} a_{ij}(k, l) (v_{ij})_{k_s, l_s} A_{kl} \quad (26)$$

Here  $a_{1\bar{3}}(1, 1) = a_{2\bar{4}}(1, 1) = -\frac{4}{3}$ ,  $a_{12}(1, 1) = a_{2\bar{3}}(1, 1) = a_{2\bar{4}}(1, 1) = a_{\bar{3}4}(1, 1) = 0$  and  $a_{12}(1, 2) = a_{\bar{3}4}(1, 2) = -a_{1\bar{3}}(1, 2) = -a_{2\bar{4}}(1, 2)$ ,  $-a_{2\bar{3}}(1, 2) = -a_{1\bar{4}}(1, 2) = \frac{4}{9}f$  and similarly others are the matrix elements of the  $\mathbf{F}_i \cdot \mathbf{F}_j$  operator in the gluonic basis  $\{|1\rangle_g, |2\rangle_g\}$ .  $A_{kl}$  are the elements of spin overlap matrix  $A$ .

$$A = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \quad (27)$$

(See appendix-a for details).

The hyperfine interactions are neglected here because they do not affect the scattering amplitude much [27]. So the potential energy matrix becomes

$$V^P = \begin{pmatrix} \frac{-4}{3}(v_{1\bar{3}} + v_{2\bar{4}})_{1s, 1s} & \frac{4}{9}f(v_{12} + v_{\bar{3}4} - v_{1\bar{3}} - v_{2\bar{4}} - v_{1\bar{4}} - v_{2\bar{3}})_{1s, 2s} \\ \frac{4}{9}f(v_{12} + v_{\bar{3}4} - v_{1\bar{3}} - v_{2\bar{4}} - v_{1\bar{4}} - v_{2\bar{3}})_{2s, 1s} & \frac{-4}{3}(v_{1\bar{4}} + v_{2\bar{3}})_{2s, 2s} \end{pmatrix} \quad (28)$$

Where the subscript  $s$  indicates the spin states.

Now we come towards kinetic energy operator. In spin space,  $K$  and  $(\sum_{i=1}^4 m_i - E_c)$  are unit operators. So their matrix elements in spin-gluonic bases are equal to matrix elements in gluonic basis multiplied by spin overlaps. i.e.

$$K_{k_{gs}, l_{gs}} = {}_{gs} \langle k | K | l \rangle_{gs} = {}_s \langle k | l \rangle_s {}_g \langle k | K | l \rangle_g = A_{kl} {}_g \langle k | K | l \rangle_g \quad (29)$$

and

$$\begin{aligned} {}_{gs} \langle k | (\sum_{i=1}^4 m_i - E_c) | l \rangle_{gs} &= (\sum_{i=1}^4 m_i - E_c) {}_{gs} \langle k | l \rangle_{gs} = (\sum_{i=1}^4 m_i - E_c) N_{k_{gs}, l_{gs}} \\ &= (\sum_{i=1}^4 m_i - E_c) {}_s \langle k | l \rangle_s {}_g \langle k | l \rangle_g = (\sum_{i=1}^4 m_i - E_c) A_{kl} {}_g \langle k | l \rangle_g \end{aligned} \quad (30)$$

where

$$N_{k_{gs}, l_{gs}} = {}_{gs} \langle k | l \rangle_{gs} \quad (31)$$

Using eqs. (31), (29) and (26) in eq. (19), we get

$$\begin{aligned} \int d^3 \mathbf{R}'_k \left[ \mathbf{K}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) + \mathbf{V}_{k_{gs}, k_{gs}}^P(\mathbf{R}_k, \mathbf{R}'_k) + \left( \sum_{i=1}^4 m_i - E_c \right) \mathbf{N}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) \right] \chi_k(\mathbf{R}'_k) + \\ \int_{l \neq k} d^3 \mathbf{R}_l \left[ \mathbf{K}_{k_{gs}, l_{gs}}(\mathbf{R}_k, \mathbf{R}_l) + \mathbf{V}_{k_{gs}, l_{gs}}^P(\mathbf{R}_k, \mathbf{R}_l) + \left( \sum_{i=1}^4 m_i - E_c \right) \mathbf{N}_{k_{gs}, l_{gs}}(\mathbf{R}_k, \mathbf{R}_l) \right] \chi_l(\mathbf{R}_l) = 0, \end{aligned} \quad (32)$$

for  $k, l = 1, 2$ .

$\mathbf{K}_{k_{gs}, l_{gs}}$ ,  $\mathbf{V}_{k_{gs}, l_{gs}}^P$  and  $\mathbf{N}_{k_{gs}, l_{gs}}$  are defined as

$$\int d^3 \mathbf{R}'_l \mathbf{K}_{k_{gs}, l_{gs}}(\mathbf{R}_k, \mathbf{R}'_l) \chi_l(\mathbf{R}'_l) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) K_{k_{gs}, l_{gs}} \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) \quad (33)$$

$$\int d^3 \mathbf{R}'_l \mathbf{V}_{k_{gs}, l_{gs}}^P(\mathbf{R}_k, \mathbf{R}'_l) \chi_l(\mathbf{R}'_l) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) V_{k_{gs}, l_{gs}}^P \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) \quad (34)$$

$$\int d^3 \mathbf{R}'_l \mathbf{N}_{k_{gs}, l_{gs}}(\mathbf{R}_k, \mathbf{R}'_l) \chi_l(\mathbf{R}'_l) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) N_{k_{gs}, l_{gs}} \chi_l(\mathbf{R}_l) \xi_l(\mathbf{y}_l) \zeta_l(\mathbf{z}_l) \quad (35)$$

### 3.1 Space Differentiations and Integrations

In this section we describe how we performed spatial integrations on right hand side of eqs. (33-35). For that we have to substitute  $K_{k_{gs}, l_{gs}}$ ,  $V_{k_{gs}, l_{gs}}^P$  and  $N_{k_{gs}, l_{gs}}$  expressions from eqs. (29), (26) and (31). As for  $K_{k_{gs}, l_{gs}}$  is concerned we have to do spacial differentiations as well because it involves kinetic energy operator. In the case of  $k = l$ ,  $R_l$  is linearly independent of  $\mathbf{y}_k$  and  $\mathbf{z}_k$  thus  $\chi_l(\mathbf{R}_l)$  can be taken out side of integration. For  $k \neq l$  we have to write  $\mathbf{R}_l$ ,  $\mathbf{y}_l$  and  $\mathbf{z}_l$  in terms of  $\mathbf{R}_k$ ,  $\mathbf{y}_k$  and  $\mathbf{z}_k$ .

First for  $k = l$  in eq. (31) by using eqs. (27 and 20) for spin and colour overlap matrix diagonal elements, we have

$$N_{k_{gs}, k_{gs}} = {}_{gs} \langle k | k \rangle_{gs} = {}_s \langle k | k \rangle_s {}_g \langle k | k \rangle_g = A_{kk} N(f)_{kk} = 1.$$

Putting this in eq. (35) we get

$$\int d^3 \mathbf{R}'_k \mathbf{N}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) \chi_k(\mathbf{R}'_k) = \chi_k(\mathbf{R}_k) \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k^2(\mathbf{y}_k) \zeta_k^2(\mathbf{z}_k). \quad (36)$$

Because  $\xi_k(\mathbf{y}_k)$  and  $\zeta_k(\mathbf{z}_k)$  are normalized, so we get

$$\int d^3 \mathbf{R}'_k \mathbf{N}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) \chi_k(\mathbf{R}'_k) = \chi_k(\mathbf{R}_k). \quad (37)$$

Or

$$\mathbf{N}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) = \delta(\mathbf{R}_k, \mathbf{R}'_k), \quad (38)$$

for  $k = 1, 2$ .

Now for kinetic energy matrix elements from eq. (29) for  $k = l$ , we have

$$K_{k_{gs}, k_{gs}} = A_{kk} {}_g \langle k | K | k \rangle_g. \quad (39)$$

$A_{kk} = 1$  from eq. (27) while from eq. (22) we have

$${}_g \langle k | K | k \rangle_g = N(f)_{k,k}^{\frac{1}{2}} \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) N(f)_{k,k}^{\frac{1}{2}} \quad (40)$$

Using eq. (20)

$${}_g \langle k | K | k \rangle_g = \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) \quad (41)$$

As the four quarks are not of the same mass in this chapter so there appears index  $i$  with  $m$ . Substituting

$$K_{k_{gs}, k_{gs}} = \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) \quad (42)$$



for  $k=1,2$ , in eq. (33) we get

$$\int d^3 \mathbf{R}'_k \mathbf{K}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) \chi_k(\mathbf{R}'_k) = \int d^3 \mathbf{y}_k d^3 \mathbf{z}_k \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) \chi_k(\mathbf{R}_k) \xi_k(\mathbf{y}_k) \zeta_k(\mathbf{z}_k) \quad (43)$$

By using eqs. (1), (2), (3) to (5), for  $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$  and for  $\mathbf{y}_k$  and  $\mathbf{z}_k$ , we can write in the set  $\{\mathbf{R}_k, \mathbf{y}_k, \mathbf{z}_k\}$

$$\left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) = -\frac{1}{2m} [s_k \nabla_{\mathbf{R}_k}^2 + q_k \nabla_{\mathbf{y}_k}^2 + t_k \nabla_{\mathbf{z}_k}^2] \quad (44)$$

for  $k = 1, 2$ ,

with  $m$  being the constituent mass of the light quark, up or down.

Here

$$s_1 = \frac{2}{r+1}, \quad q_1 = t_1 = \frac{r+1}{r}, \quad s_2 = \frac{r+1}{2r}, \quad q_2 = 2, \quad t_2 = \frac{2}{r}. \quad (45)$$

. By using eq. (44) in to eq. (43) and doing all the space differentiations and integrations, we get

$$\mathbf{K}_{k_{gs}, k_{gs}}(\mathbf{R}_k, \mathbf{R}'_k) = \delta(\mathbf{R}_k, \mathbf{R}'_k) \left[ \frac{3}{4} (\lambda_{k1} + \lambda_{k2}) - \frac{s_k}{2m} \nabla_{\mathbf{R}_k}^2 \right] \quad (46)$$

for  $k = 1, 2$ . Here

$$\lambda_{k1} = \frac{q_k}{2md_{k1}^2} \quad \text{and} \quad \lambda_{k2} = \frac{t_k}{2md_{k2}^2}. \quad (47)$$

For the potential energy matrix, from eq. (26) and eq. (27) for  $k = l$ , we have

$$V_{k_{gs}, k_{gs}}^P = \sum_{i < j} a_{ij}(k, k) (v_{ij})_{k_s, k_s} \quad (48)$$

The only non zero coefficients  $a_{ij}(k, k)$  for  $k = l = 1$  are  $a_{13}$  and  $a_{14}$  and for  $k = l = 2$  are  $a_{14}$  and  $a_{23}$ . Using the quadratic form of potential from eq. (12), eq. (48) becomes

$$V_{k_{gs}, k_{gs}}^P = -\frac{4}{3} [2\bar{C} + C\mathbf{y}_k^2 + C\mathbf{z}_k^2] \quad (49)$$

Using this in eq. (34) we get the result

$$\mathbf{V}_{k_{gs}, k_{gs}}^P = \delta(\mathbf{R}_k, \mathbf{R}'_k) \left[ -\frac{8}{3} \bar{C} - 4C[d_{k1}^2 + d_{k2}^2] \right] \quad (50)$$

For writing the off-diagonal elements first we write  $\mathbf{y}_l, \mathbf{z}_l$  and  $\mathbf{R}_l$  as linear combination of  $\mathbf{y}_k, \mathbf{z}_k$  and  $\mathbf{R}_k$ . This means, we use

$$\mathbf{R}_2 = \frac{\mathbf{y}_1 - \mathbf{z}_1}{2} \quad (51)$$

$$\mathbf{R}_3 = -\left(\frac{r-1}{r+1}\right) \mathbf{R}_1 + \frac{2r}{(r+1)^2} (\mathbf{y}_1 + \mathbf{z}_1) \quad (52)$$

$$\mathbf{R}_1 = \frac{\mathbf{y}_2 - r\mathbf{z}_2}{r+1} \quad (53)$$

$$\mathbf{R}_3 = \frac{\mathbf{y}_2 + r\mathbf{z}_2}{r+1} \quad (54)$$

$$\mathbf{R}_1 = -\left(\frac{r-1}{r+1}\right) \mathbf{R}_3 + \frac{2r}{(r+1)^2} (\mathbf{y}_3 + \mathbf{z}_3) \quad (55)$$

$$\mathbf{R}_2 = \frac{\mathbf{y}_3 - \mathbf{z}_3}{2} \quad (56)$$

It is clear from the above relations that the integration variables  $\mathbf{y}_k$  and  $\mathbf{z}_k$  are not appropriate for the integrations to do next. So, we have to replace them by their combinations with one being identical to  $\mathbf{R}_l$  and other independent of it. Because of different quark masses now we can not write e.g.  $\mathbf{R}_3$  simply as linear combination of  $\mathbf{y}_1$  and  $\mathbf{z}_1$ . Actually in place of  $\mathbf{R}_1$ ,  $\mathbf{y}_1$  and  $\mathbf{z}_1$ , we used

$$\mathbf{f}_1 = \mathbf{R}_3 = -\left(\frac{r-1}{r+1}\right)\mathbf{R}_1 + \frac{2r}{(r+1)^2}(\mathbf{y}_1 + \mathbf{z}_1) \quad (57)$$

$$\mathbf{g}_1 = -\left(\frac{r-1}{r+1}\right)\mathbf{R}_1 - \frac{2r}{(r+1)^2}(\mathbf{y}_1 + \mathbf{z}_1) \quad (58)$$

$$\mathbf{h}_1 = \mathbf{R}_2 = \frac{\mathbf{y}_1 - \mathbf{z}_1}{2} \quad (59)$$

$\mathbf{R}_2$ ,  $\mathbf{y}_2$  and  $\mathbf{z}_2$  are replaced by

$$\mathbf{f}_2 = \mathbf{R}_2 \quad (60)$$

$$\mathbf{g}_2 = \mathbf{R}_3 = \frac{\mathbf{y}_2 + r\mathbf{z}_2}{r+1} \quad (61)$$

$$\mathbf{h}_2 = \mathbf{R}_1 = \frac{\mathbf{y}_2 - r\mathbf{z}_2}{r+1} \quad (62)$$

Similarly for  $\mathbf{R}_3$ ,  $\mathbf{y}_3$  and  $\mathbf{z}_3$  we have

$$\mathbf{f}_3 = \mathbf{R}_1 = -\left(\frac{r-1}{r+1}\right)\mathbf{R}_3 + \frac{2r}{(r+1)^2}(\mathbf{y}_3 + \mathbf{z}_3) \quad (63)$$

$$\mathbf{g}_3 = -\left(\frac{r-1}{r+1}\right)\mathbf{R}_3 - \frac{2r}{(r+1)^2}(\mathbf{y}_3 + \mathbf{z}_3) \quad (64)$$

$$\mathbf{h}_3 = \mathbf{R}_2 = \frac{\mathbf{y}_3 - \mathbf{z}_3}{2} \quad (65)$$

With these new integration variables  $\mathbf{f}_k$ ,  $\mathbf{g}_k$  and  $\mathbf{h}_k$  (or their combinations  $\mathbf{f}_k + \mathbf{g}_k$  and  $\mathbf{f}_k - \mathbf{g}_k$ ) for  $k = 1, 2$  we can write expression for  $\mathbf{N}_{k_{gs}, l_{gs}}$  and other kernels present on L.H.S. of eqs. (33-35). First of all we write expression for  $N_{k_{gs}, l_{gs}}$  from eq. (31) with the help of eqs. (27 and 20)

$$N_{k_{gs}, l_{gs}} = \frac{f}{3} A_{kl} \quad (66)$$

for  $k \neq l$ . Substituting in eq. (35) expressing all the vectors there in terms of  $\mathbf{f}_k$ ,  $\mathbf{g}_k$  and  $\mathbf{h}_k$  and doing all the integrations other than  $\mathbf{R}_l$  we get

$$\mathbf{N}_{1_{gs}, 2_{gs}} = \frac{l_o}{3\sqrt{2}} \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2) \quad (67)$$

and

$$\mathbf{N}_{2_{gs}, 1_{gs}} = \mathbf{N}_{1_{gs}, 2_{gs}} \quad (68)$$

Here

$$l_o = (r+1)^{\frac{9}{4}} r^{\frac{-15}{8}} 2^{\frac{3}{4}} (\pi \alpha d^2)^{\frac{-3}{2}} \quad (69)$$

$$l_1 = \frac{1}{4d^2} \left( \frac{r+1}{2} \right)^2 \left[ \gamma - \frac{\beta^2}{\alpha} \right] \quad (70)$$

$$l_2 = 4\bar{k} + \frac{1}{2d^2} \sqrt{\frac{2r}{r+1}} \quad (71)$$

and

$$\alpha = 8\bar{k}d^2 \left[ \frac{r^2+1}{r^2} \right] + 1 + r^{\frac{-3}{2}} \left[ \frac{(r+1)^2}{\sqrt{2(r+1)}} + 1 \right] \quad (72)$$

$$\beta = 8\bar{k}d^2 \left[ \frac{r^2-1}{r^2} \right] + 1 + r^{\frac{-3}{2}} \left[ \frac{r^2-1}{\sqrt{2(r+1)}} - 1 \right] \quad (73)$$

$$\gamma = 8\bar{k}d^2 \left[ \frac{r^2+1}{r^2} \right] + 1 + r^{\frac{-3}{2}} \left[ \frac{(r-1)^2}{\sqrt{2(r+1)}} + 1 \right] \quad (74)$$

The above integrations leave the kernels as functions of  $\mathbf{f}_k$ ,  $\mathbf{g}_k$  and  $\mathbf{h}_k$  which are then replaced by  $\mathbf{R}_k$  and  $\mathbf{R}_l$  using eqs. (57-65). E.g. for  $\mathbf{N}_{1gs,2gs}$ ,  $\mathbf{f}_1 - \mathbf{g}_1$  is integrated out leaving with

$$\mathbf{f}_1 + \mathbf{g}_1 = -2 \left( \frac{r-1}{r+1} \right) \mathbf{R}_1$$

and

$$\mathbf{h}_1 = \mathbf{R}_2$$

etc.

Now for the kinetic energy kernel for  $k \neq l$ . Here along with integration firstly we have to do space differentiations as well. From eq. (29) using eq. (27) and eq. (22), we have

$$K_{kgs,lgs} = \frac{1}{3\sqrt{2}} f^{\frac{1}{2}} \left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) f^{\frac{1}{2}}. \quad (75)$$

We have expressed  $\left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right)$  in eq. (44) in term of  $\nabla_{\mathbf{R}_k}^2$ ,  $\nabla_{\mathbf{y}_k}^2$  and  $\nabla_{\mathbf{z}_k}^2$ . Now using relations from eqs. (51-65), we can write for  $k=1$

$$\left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) = -\frac{1}{2m} \left[ \frac{8(r-1)^2}{(r+1)^3} \nabla_{(\mathbf{f}_1+\mathbf{g}_1)}^2 + \frac{32r}{(r+1)^3} \nabla_{(\mathbf{f}_1-\mathbf{g}_1)}^2 + \frac{r+1}{2r} \nabla_{\mathbf{h}_1}^2 \right], \quad (76)$$

and for  $k=2$

$$\left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) = -\frac{1}{2m} \left[ \frac{r+1}{2r} \nabla_{\mathbf{f}_2}^2 + \frac{8}{(r+1)^2} \left( \nabla_{(\mathbf{g}_2+\mathbf{h}_2)}^2 + r \nabla_{(\mathbf{g}_2-\mathbf{h}_2)}^2 \right) \right].$$

Or

$$\left( \sum_{i=1}^4 -\frac{\nabla_i^2}{2m_i} \right) = -\frac{1}{2m} \left[ \frac{r+1}{2r} \nabla_{\mathbf{R}_2}^2 + \frac{8}{(r+1)^2} \left( \nabla_{(\mathbf{R}_3+\mathbf{R}_1)}^2 + r \nabla_{(\mathbf{R}_3-\mathbf{R}_1)}^2 \right) \right]. \quad (77)$$

Substituting eqs. (77) and (75) in eq. (33), we get

$$\mathbf{K}_{1gs,2gs} = -\frac{l_o}{2m} \frac{1}{3\sqrt{2}} \left[ r_{11} \mathbf{R}_1^2 + r_{12} \mathbf{R}_2^2 + r_{10} \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2) \quad (78)$$

and

$$\mathbf{K}_{2gs,1gs} = -\frac{l_o}{2m} \frac{1}{3\sqrt{2}} \left[ r_{21} \mathbf{R}_1^2 + r_{22} \mathbf{R}_2^2 + r_{20} \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2), \quad (79)$$

where

$$r_{11} = \left(\frac{r+1}{2}\right)^4 \left[ \frac{8(r-1)^2}{(r+1)^3} \left\{ \left(\frac{r-1}{r+1}\right) \left( \frac{8\bar{k}}{(r-1)^2} + \frac{1+\sqrt{r}}{(r-1)^2 d^2} \right) - \left( \frac{\beta}{\alpha} - \frac{r-1}{r+1} \right) \left( \frac{2\bar{k}}{r} + \frac{1}{2d^2\sqrt{r}(1+\sqrt{r})} \right) \right\}^2 + \frac{32r}{(r+1)^3} \left\{ \left(\frac{r-1}{r+1}\right) \left( \frac{2\bar{k}}{r} + \frac{1}{2d^2\sqrt{r}(1+\sqrt{r})} \right) - \left( \frac{\beta}{\alpha} - \frac{r-1}{r+1} \right) \left( \bar{k} \frac{r^2+1}{r^2} + \frac{r^{-3/2}+1}{4d^2} \right) \right\}^2 \right], \quad (80)$$

$$r_{10} = -\frac{3}{2} \left(\frac{r+1}{2}\right)^2 \left\{ 8 \frac{(r-1)^2}{(r+1)^3} \left( \frac{8\bar{k}}{(r-1)^2} + \frac{1+\sqrt{r}}{(r-1)^2 d^2} \right) + \frac{32r}{(r+1)^3} \left( \frac{\bar{k}(r^2+1)}{r^2} + \frac{r^{-3/2}+1}{4d^2} \right) \right\} + \frac{3}{2} \frac{d^2}{\alpha} (r+1)^2 \left\{ \frac{8(r-1)^2}{(r+1)^3} \left( \frac{2\bar{k}}{r} + \frac{1}{2d^2\sqrt{r}(1+\sqrt{r})} \right)^2 + \frac{32r}{(r+1)^3} \left( \frac{\bar{k}(r^2+1)}{r^2} + \frac{r^{-3/2}+1}{4d^2} \right)^2 \right\} - \frac{6(r+1)}{2r} \left( 2\bar{k} + \frac{1}{2d^2} \sqrt{\frac{2r}{r+1}} \right), \quad (81)$$

$$r_{12} = 4 \left( \frac{r+1}{2r} \right) \left( 2\bar{k} + \frac{1}{2d^2} \sqrt{\frac{2r}{r+1}} \right)^2, \quad (82)$$

$$r_{22} = r_{12}, \quad (83)$$

$$r_{20} = \frac{8}{(r+1)^2} \left(\frac{r+1}{2}\right)^2 \frac{24d^2}{\alpha} \left\{ \left( \bar{k} + \frac{1}{4d^2} \right)^2 + r \left( \frac{\bar{k}}{r^2} + \frac{r^{-3/2}}{4d^2} \right)^2 \right\} - 6 \frac{8}{(r+1)^2} \left(\frac{r+1}{2}\right)^2 \left\{ \left( \bar{k} + \frac{1}{4d^2} \right) + r \left( \frac{\bar{k}}{r^2} + \frac{r^{-3/2}}{4d^2} \right) \right\} - 6 \frac{r+1}{2r} \left\{ 2\bar{k} + \frac{1}{2d^2} \sqrt{\frac{2r}{r+1}} \right\} \quad (84)$$

and

$$r_{21} = 2(r+1)^2 \left\{ \left( 1 - \frac{\beta}{\alpha} \right)^2 \left( \bar{k} + \frac{1}{4d^2} \right)^2 + r \left( 1 + \frac{\beta}{\alpha} \right)^2 \left( \frac{\bar{k}}{r^2} + \frac{r^{-3/2}}{4d^2} \right)^2 \right\}. \quad (85)$$

Now for the potential energy kernel using eq. (28), eq. (12), and eq. (27) in eq. (26), we get for  $k \neq l$

$$V_{1_{gs}, 2_{gs}}^P = -\frac{8}{3} \bar{C} \frac{f}{3\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{4}{9} f C \left[ \mathbf{y}_3^2 + \mathbf{z}_3^2 - \mathbf{y}_1^2 - \mathbf{z}_1^2 - \mathbf{y}_2^2 - \mathbf{z}_2^2 \right] \quad (86)$$

and

$$V_{2_{gs}, 1_{gs}}^P = V_{1_{gs}, 2_{gs}}^P. \quad (87)$$

Putting this expression in eq. (34), changing variables and doing all the integrations, we get

$$\mathbf{V}_{k_{gs}, l_{gs}}^P = -\frac{8}{3} \bar{C} \mathbf{N}_{k_{gs}, l_{gs}} + A_{kl} \mathbf{V}_{k_{gs}, l_{gs}}, \quad (88)$$

where

$$\mathbf{V}_{1_{gs}, 2_{gs}} = \mathbf{V}_{2_{gs}, 1_{gs}} = l_o \left[ n_1 \mathbf{R}_1^2 + n_o \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2). \quad (89)$$

$\mathbf{N}_{k_{gs}, l_{gs}}$  is defined in eq. (67) and eq. (68). Here

$$n_o = -\frac{8}{3} C \left( \frac{r+1}{r} \right)^2 \frac{d^2}{\alpha} \quad (90)$$

and

$$n_1 = -\frac{4}{9} C \left\{ \frac{(r+1)^4}{4r^2} \right\} \left( \frac{\beta}{\alpha} - \frac{r-1}{r+1} \right)^2. \quad (91)$$

## 4 Writing Coupled Equations

For  $k = 1$  putting expressions from eqs. (46), (50), (38), (78), (88), (89) and (67) in eq. (32), we get

$$\begin{aligned} & \left[ \frac{3}{4}(\lambda_{11} + \lambda_{12}) - \frac{s_1}{2m} \nabla_{\mathbf{R}_1}^2 - \frac{8}{3} \bar{C} - 4C[d_{11}^2 + d_{12}^2] + 2m(r+1) - E_c \right] \chi_1(\mathbf{R}_1) + \\ & \sqrt{2} l_o \int d^3 \mathbf{R}_2 \left[ -\frac{1}{2m} \frac{1}{6} [r_{11} \mathbf{R}_1^2 + r_{12} \mathbf{R}_2^2 + r_{10}] + \frac{1}{2} [n_1 \mathbf{R}_1^2 + n_o] \right. \\ & \left. - \frac{1}{6} \left( E_c + \frac{8}{3} \bar{C} - 2m(r+1) \right) \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2) \chi_2(\mathbf{R}_2) = 0 \end{aligned} \quad (92)$$

Now for  $k = 2$  putting expressions from eqs. (46), (50), (38), (79), (88), (89) and (68) in (32), we get

$$\begin{aligned} & \left[ \frac{3}{4}(\lambda_{21} + \lambda_{22}) - \frac{s_2}{2m} \nabla_{\mathbf{R}_2}^2 - \frac{8}{3} \bar{C} - 4C[d_{21}^2 + d_{22}^2] + 2m(r+1) - E_c \right] \chi_2(\mathbf{R}_2) + \\ & \sqrt{2} l_o \int d^3 \mathbf{R}_1 \left[ -\frac{1}{2m} \frac{1}{6} [r_{21} \mathbf{R}_1^2 + r_{22} \mathbf{R}_2^2 + r_{20}] + \frac{1}{2} [n_1 \mathbf{R}_1^2 + n_o] \right. \\ & \left. - \frac{1}{6} \left( E_c + \frac{8}{3} \bar{C} - 2m(r+1) \right) \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2) \chi_1(\mathbf{R}_1) = 0 \end{aligned} \quad (93)$$

It is clear from eqs. (92) and (93) that off-diagonal parts vanishes for large values of  $\mathbf{R}_1$  and  $\mathbf{R}_2$ . So in this limit the two equations become

$$\begin{aligned} & \left[ \left( \frac{3}{4} \lambda_{11} - \frac{4}{3} \bar{C} - 4C d_{11}^2 + m(r+1) \right) - \left( \frac{s_1}{2m} \nabla_{\mathbf{R}_1}^2 + E_c \right) \right. \\ & \left. + \left( \frac{3}{4} \lambda_{12} - \frac{4}{3} \bar{C} - 4C d_{12}^2 + m(r+1) \right) \right] \chi_1(\mathbf{R}_1) \longrightarrow_{R_1 \rightarrow \infty} 0 \end{aligned} \quad (94)$$

and

$$\begin{aligned} & \left[ \left( \frac{3}{4} \lambda_{21} - \frac{4}{3} \bar{C} - 4C d_{21}^2 + 2m \right) - \left( \frac{s_2}{2m} \nabla_{\mathbf{R}_2}^2 + E_c \right) \right. \\ & \left. + \left( \frac{3}{4} \lambda_{22} - \frac{4}{3} \bar{C} - 4C d_{22}^2 + 2m_c \right) \right] \chi_2(\mathbf{R}_2) \longrightarrow_{R_2 \rightarrow \infty} 0. \end{aligned} \quad (95)$$

For the large separation there is no interaction between the two mesons. So the total center of mass energy in the large separation limit will be the sum of kinetic energy of the relative motion of mesons and masses of the two mesons. i.e. for  $R_1 \rightarrow \infty$ , we have

$$\left[ -\frac{1}{2\mu_{D^o \bar{D}^{o*}}} \nabla_{\mathbf{R}_1}^2 + M_D + M_{\bar{D}^{o*}} \right] \chi_1(\mathbf{R}_1) = E_c \chi_1(\mathbf{R}_1), \quad (96)$$

and for  $R_2 \rightarrow \infty$ , we have

$$\left[ -\frac{1}{2\mu_{\omega J/\psi}} \nabla_{\mathbf{R}_2}^2 + M_\omega + M_{J/\psi} \right] \chi_2(\mathbf{R}_2) = E_c \chi_2(\mathbf{R}_2). \quad (97)$$

By making a comparison of eq. (94) and eq. (96), we can write eq. (92) as follow

$$\begin{aligned} & \left[ M_D + M_{\bar{D}^{o*}} - \frac{1}{2\mu_{D^o \bar{D}^{o*}}} \nabla_{\mathbf{R}_1}^2 - E_c \right] \chi_1(\mathbf{R}_1) + \sqrt{2} l_o \int d^3 \mathbf{R}_2 \left[ -\frac{1}{2m} \frac{1}{6} [r_{11} \mathbf{R}_1^2 + r_{12} \mathbf{R}_2^2 + r_{10}] \right. \\ & \left. + \frac{1}{2} [n_1 \mathbf{R}_1^2 + n_o] - \frac{1}{6} \left( E_c + \frac{8}{3} \bar{C} - 2m(r+1) \right) \right] \exp(-l_1 \mathbf{R}_1^2 - l_2 \mathbf{R}_2^2) \chi_2(\mathbf{R}_2) = 0. \end{aligned} \quad (98)$$

Similarly by making a comparison of eq. (95) and eq. (97), we can write eq. (93) as follows

$$\begin{aligned} & \left[ M_\omega + M_{J/\psi} - \frac{1}{2\mu_{\omega J/\psi}} \nabla_{\mathbf{R}_2}^2 - E_c \right] \chi_2(\mathbf{R}_2) + \sqrt{2}l_o \int d^3\mathbf{R}_1 \left[ -\frac{1}{2m} \frac{1}{6} \left[ r_{21}\mathbf{R}_1^2 + r_{22}\mathbf{R}_2^2 + r_{20} \right] \right. \\ & \left. + \frac{1}{2} \left[ n_1\mathbf{R}_1^2 + n_o \right] - \frac{1}{6} \left( E_c + \frac{8}{3}\bar{C} - 2m(r+1) \right) \right] \exp(-l_1\mathbf{R}_1^2 - l_2\mathbf{R}_2^2) \chi_1(\mathbf{R}_1) = 0. \end{aligned} \quad (99)$$

By taking Fourier transform of eq. (98) and (99), we get

$$\begin{aligned} & \left[ M_D + M_{\bar{D}^{o*}} + \frac{1}{2\mu_{D^o\bar{D}^{o*}}} \mathbf{P}_1^2 - E_c \right] \chi_1(\mathbf{P}_1) + \left[ -\frac{1}{2m} \frac{r_{11}}{6} + \frac{n_1}{2} \right] A_2(l_2) F_b(\mathbf{P}_1, l_1) + \\ & \left[ -\frac{1}{2m} \frac{r_{12}}{6} B_2(l_2) + \left( -\frac{1}{2m} \frac{r_{10}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right) A_2(l_2) \right] F_a(\mathbf{P}_1, l_1) = 0, \end{aligned} \quad (100)$$

and

$$\begin{aligned} & \left[ M_\omega + M_{J/\psi} + \frac{1}{2\mu_{\omega J/\psi}} \mathbf{P}_2^2 - E_c \right] \chi_2(\mathbf{P}_2) - \frac{1}{2m} \frac{r_{22}}{6} A_1(l_1) F_b(\mathbf{P}_2, l_2) + \\ & \left[ \left( -\frac{1}{2m} \frac{r_{20}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right) A_1(l_1) + \left( -\frac{1}{2m} \frac{r_{21}}{6} + \frac{n_1}{2} \right) B_1(l_1) \right] F_a(\mathbf{P}_2, l_2) = 0. \end{aligned} \quad (101)$$

Here

$$\chi_k(\mathbf{P}_k) = \int \frac{d^3\mathbf{R}_k}{(2\pi)^{\frac{3}{2}}} \exp[i\mathbf{P}_k \cdot \mathbf{R}_k] \chi_k(\mathbf{R}_k), \quad (102)$$

$$E'_c = E_c + \frac{8}{3}\bar{C} - 2m(r+1), \quad (103)$$

$$A_k(u) = l'_o \int d^3\mathbf{R}_k \exp[-u\mathbf{R}_k^2] \chi_k(\mathbf{R}_k), \quad (104)$$

Where

$$l'_o = \sqrt{2}l_o,$$

$$B_k(u) = l'_o \int d^3\mathbf{R}_k \exp[-u\mathbf{R}_k^2] \mathbf{R}_k^2 \chi_k(\mathbf{R}_k), \quad (105)$$

$$F_a(\mathbf{P}_k, u) = \int \frac{d^3\mathbf{R}_k}{(2\pi)^{\frac{3}{2}}} \exp[i\mathbf{P}_k \cdot \mathbf{R}_k] \exp[-u\mathbf{R}_k^2], \quad (106)$$

$$F_b(\mathbf{P}_k, u) = \int \frac{d^3\mathbf{R}_k}{(2\pi)^{\frac{3}{2}}} \exp[i\mathbf{P}_k \cdot \mathbf{R}_k] \mathbf{R}_k^2 \exp[-u\mathbf{R}_k^2], \quad (107)$$

$$F_a(\mathbf{P}_k, u) = \frac{1}{(2\pi)^{\frac{3}{2}}} \exp \left[ -\frac{\mathbf{P}_k^2}{4u} \right] \quad (108)$$

and

$$F_b(\mathbf{P}_k, u) = F_a(\mathbf{P}_k, u) \left[ \frac{1}{2u} \right] \left[ 3 - \frac{\mathbf{P}_k^2}{2u} \right], \quad (109)$$

for  $k = 1, 2$ .

For the incoming waves in the first channel eq. (100) and (101) can be formally solved [40] as

$$\begin{aligned} \chi_1(p_1) = & \frac{\delta(p_1 - p_c(1))}{p_c^2(1)} - \frac{F_b(p_1, l_1)}{\Delta_1(p_1)} \left[ -\frac{1}{2m} \frac{r_{11}}{6} + \frac{n_1}{2} \right] A_2(l_2) - \\ & \frac{F_a(p_1, l_1)}{\Delta_1(p_1)} \left[ -\frac{1}{2m} \frac{r_{12}}{6} B_2(l_2) + \left( -\frac{1}{2m} \frac{r_{10}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right) A_2(l_2) \right] = 0 \end{aligned} \quad (110)$$

and

$$\begin{aligned} \chi_2(p_2) = & \frac{F_b(p_2, l_2)}{\Delta_2(p_2)} \frac{1}{2m} \frac{r_{22}}{6} A_1(l_1) - \frac{F_a(p_2, l_2)}{\Delta_2(p_2)} \left[ \left( -\frac{1}{2m} \frac{r_{20}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right) \right. \\ & \left. A_1(l_1) + \left( -\frac{1}{2m} \frac{r_{21}}{6} + \frac{n_1}{2} \right) B_1(l_1) \right] = 0, \end{aligned} \quad (111)$$

where

$$\Delta_1(p_1) = \frac{p_1^2}{2\mu_{D^o \bar{D}^{o*}}} + M_D + M_{\bar{D}^{o*}} - E_c - \iota\epsilon, \quad (112)$$

$$\Delta_2(p_2) = \frac{p_2^2}{2\mu_{\omega J/\psi}} + M_\omega + M_{J/\psi} - E_c - \iota\epsilon, \quad (113)$$

$$p_c(1) = \sqrt{2\mu_{D^o \bar{D}^{o*}}(E_c - M_D - M_{\bar{D}^{o*}})} \quad (114)$$

and

$$p_c(2) = \sqrt{2\mu_{\omega J/\psi}(E_c - M_\omega - M_{J/\psi})}. \quad (115)$$

Because of the spherical symmetry of the S-wave ( $l = 0$ ) here,  $\mathbf{P}_i$  is replaced with  $p_i$  (magnitude) with  $i = 1, 2$ . Using the Parseval relation corresponding to eqs. (104) and (105), we have

$$A_k(u) = 4\pi l'_o \int dp_k p_k^2 F_a(p_k, u) \chi_k(p_k) \quad (116)$$

and

$$B_k(u) = 4\pi l'_o \int dp_k p_k^2 F_b(p_k, u) \chi_k(p_k), \quad (117)$$

for  $k = 1, 2$ .

Multiplying eq. (110) by  $4\pi p_1^2 F_a(p_1, l_1)$  and integrating w.r.t.  $p_1$  and with using eq. (116) we get

$$\frac{A_1(l_1)}{l'_o} = 4\pi F_a(p_c(1), l_1) - A_2(l_2) W_{11}^{(1)} - B_2(l_2) W_{12}^{(1)} \quad (118)$$

where

$$W_{11}^{(1)} = \left[ -\frac{1}{2m} \frac{r_{11}}{6} + \frac{n_1}{2} \right] J'_1(l_1, l_1) + \left[ -\frac{1}{2m} \frac{r_{10}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right] J_1(l_1, l_1), \quad (119)$$

$$W_{12}^{(1)} = -\frac{1}{2m} \frac{r_{12}}{6} J_1(l_1, l_1), \quad (120)$$

$$J_k(u, v) = 4\pi \int dp_k p_k^2 \frac{F_a(p_k, u) F_a(p_k, v)}{\Delta_k(p_k)} \quad (121)$$

and

$$J'_k(u, v) = 4\pi \int dp_k p_k^2 \frac{F_a(p_k, u) F_b(p_k, v)}{\Delta_k(p_k)}. \quad (122)$$

Similarly multiplying eq. (110) by  $4\pi p_1^2 F_b(p_1, l_1)$  and integrating w.r.t.  $p_1$  and using eq. (117), we get

$$\frac{B_1(l_1)}{l_o'} = 4\pi F_b(p_c(1), l_1) - A_2(l_2)W_{21}^{(1)} - B_2(l_2)W_{22}^{(1)}, \quad (123)$$

where

$$W_{21}^{(1)} = \left[ -\frac{1}{2m} \frac{r_{11}}{6} + \frac{n_1}{2} \right] J_1''(l_1, l_1) + \left[ -\frac{1}{2m} \frac{r_{10}}{6} + \frac{n_o}{2} - \frac{E_c'}{6} \right] J_1'(l_1, l_1), \quad (124)$$

$$W_{22}^{(1)} = -\frac{1}{2m} \frac{r_{12}}{6} J_1'(l_1, l_1), \quad (125)$$

and

$$J_k''(u, v) = 4\pi \int dp_k p_k^2 \frac{F_b(p_k, u) F_b(p_k, v)}{\Delta_k(p_k)}. \quad (126)$$

In the same way multiplying eq. (111) by  $4\pi p_2^2 F_a(p_2, l_2)$  and  $4\pi p_2^2 F_b(p_2, l_2)$  and integrating w.r.t.  $p_2$  and using eqs. (116 and 117), we get

$$\frac{A_2(l_2)}{l_o'} = -A_1(l_1)W_{11}^{(2)} - B_1(l_1)W_{12}^{(2)} \quad (127)$$

and

$$\frac{B_2(l_2)}{l_o'} = -A_1(l_1)W_{21}^{(2)} - B_1(l_1)W_{22}^{(2)}, \quad (128)$$

where

$$W_{11}^{(2)} = -\frac{1}{2m} \frac{r_{22}}{6} J_2'(l_2, l_2) + \left[ -\frac{1}{2m} \frac{r_{20}}{6} + \frac{n_o}{2} - \frac{E_c'}{6} \right] J_2(l_2, l_2), \quad (129)$$

$$W_{12}^{(2)} = \left[ -\frac{1}{2m} \frac{r_{21}}{6} + \frac{n_1}{2} \right] J_2(l_2, l_2), \quad (130)$$

$$W_{21}^{(2)} = -\frac{1}{2m} \frac{r_{22}}{6} J_2''(l_2, l_2) + \left[ -\frac{1}{2m} \frac{r_{20}}{6} + \frac{n_o}{2} - \frac{E_c'}{6} \right] J_2'(l_2, l_2) \quad (131)$$

and

$$W_{22}^{(2)} = \left[ -\frac{1}{2m} \frac{r_{21}}{6} + \frac{n_1}{2} \right] J_2'(l_2, l_2). \quad (132)$$

Equations (118), (123), (127) and (128) can be written in the matrix form as follows

$$WV_1 = V_2, \quad (133)$$

with

$$W = \begin{pmatrix} l_o'^{-1} & 0 & W_{11}^{(1)} & W_{12}^{(1)} \\ 0 & l_o'^{-1} & W_{21}^{(1)} & W_{22}^{(1)} \\ W_{11}^{(2)} & W_{12}^{(2)} & l_o'^{-1} & 0 \\ W_{21}^{(2)} & W_{22}^{(2)} & 0 & l_o'^{-1} \end{pmatrix}, \quad (134)$$

$$V_1 = \begin{pmatrix} A_1(l_1) \\ B_1(l_1) \\ A_2(l_2) \\ B_2(l_2) \end{pmatrix} \quad (135)$$



and

$$V_2 = 4\pi \begin{pmatrix} F_a(p_c(1), l_1) \\ F_b(p_c(1), l_1) \\ 0 \\ 0 \end{pmatrix}. \quad (136)$$

From eq. (133), we can have

$$V_1 = W^{-1}V_2, \quad (137)$$

which gives values of  $A_1(l_1)$ ,  $B_1(l_1)$ ,  $A_2(l_2)$  and  $B_2(l_2)$ . So eq. (110) and (111) become

$$\chi_1(p_1) = \frac{\delta(p_1 - p_c(1))}{p_c^2(1)} - \frac{1}{\Delta_1(p_1)} \left[ W_1^{(1)} A_2(l_2) - W_2^{(1)} B_2(l_2) \right] \quad (138)$$

and

$$\chi_2(p_2) = -\frac{1}{\Delta_2(p_2)} \left[ W_1^{(2)} A_1(l_1) - W_2^{(2)} B_1(l_1) \right], \quad (139)$$

where

$$W_1^{(1)} = \left[ -\frac{1}{2m} \frac{r_{11}}{6} + \frac{n_1}{2} \right] F_b(p_c(1), l_1) + \left[ -\frac{1}{2m} \frac{r_{10}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right] F_a(p_c(1), l_1), \quad (140)$$

$$W_2^{(1)} = -\frac{1}{2m} \frac{r_{12}}{6} F_a(p_c(1), l_1), \quad (141)$$

$$W_1^{(2)} = -\frac{1}{2m} \frac{r_{22}}{6} F_b(p_c(2), l_2) + \left[ -\frac{1}{2m} \frac{r_{20}}{6} + \frac{n_o}{2} - \frac{E'_c}{6} \right] F_a(p_c(2), l_2) \quad (142)$$

and

$$W_2^{(2)} = \left[ -\frac{1}{2m} \frac{r_{21}}{6} + \frac{n_1}{2} \right] F_a(p_c(2), l_2). \quad (143)$$

From eqs. (138) and (139) we can read off the T-matrix elements  $T_{11}$  and  $T_{21}$  [40]. These are proportional to the co-efficient of Green's function operators  $-\frac{1}{\Delta_1(p_1)}$  and  $-\frac{1}{\Delta_2(p_2)}$ . Here the relation between s-matrix and T-matrix is

$$s = 1 - 2iT \quad (144)$$

when both the channels are open. Eq. (144) gives

$$T_{11} = 2\mu_{D^o\overline{D}^{o*}} \frac{\pi}{2} p_c(1) \left[ W_1^{(1)} A_2(l_2) + W_2^{(1)} B_2(l_2) \right] \quad (145)$$

and

$$T_{21} = 2\mu_{\omega J\psi} \frac{\pi}{2} p_c(1) \sqrt{\frac{v_2}{v_1}} \left[ W_1^{(2)} A_1(l_1) + W_2^{(2)} B_1(l_1) \right] \quad (146)$$

where,

$$v_1 = \frac{p_c(1)}{\mu_{D^o\overline{D}^{o*}}}$$

and

$$v_2 = \frac{p_c(2)}{\mu_{\omega J\psi}}.$$

Similarly  $T_{22}$  and  $T_{12}$  can be found, using

$$V_2 = 4\pi \begin{pmatrix} 0 \\ 0 \\ F_a(p_c(2), l_2) \\ F_b(p_c(2), l_2) \end{pmatrix} \quad (147)$$

in eq. (133), for incoming waves in the 2nd channel. These become

$$T_{22} = 2\mu_{\omega J\psi} \frac{\pi}{2} p_c(2) \left[ W_1^{(2)} A_1(l_1) + W_2^{(2)} B_1(l_1) \right] \quad (148)$$

and

$$T_{12} = 2\mu_{D^0 \bar{D}^{0*}} \frac{\pi}{2} p_c(2) \sqrt{\frac{v_1}{v_2}} \left[ W_1^{(1)} A_2(l_2) + W_2^{(1)} B_2(l_2) \right]. \quad (149)$$

#### 4.1 Fitting the parameters Used

By making a comparison of eq. (97) with eq. (95), we have

$$M_{J/\psi} = \frac{3}{4} \lambda_{22} - \frac{4}{3} \bar{C} - 4C d_{22}^2 + 2m_c \quad (150)$$

From  $-4C/3 = \mu_{c\bar{c}} \lambda_{22}^2/2$ . Also reduced mass for a pair of *charm* and *anti-charm* quarks  $\mu_{c\bar{c}} = m_c/2$  because *c*-quark and  $\bar{c}$ -quark have the same constitutional mass. Thus  $-4C/3 = m_c \lambda_{22}^2/4$ , along with  $\lambda_{22} = 1/m_c d_{22}^2$  gives

$$-4C/3 = \lambda_{22}/4d_{22}^2 \quad \text{or} \quad -4C d_{22}^2 = 3\lambda_{22}/4 \quad (151)$$

We have fixed the parameters  $\omega$  and  $\bar{C}$  by spin averaging of masses of charmonium in the state 1S and the state 2S.

From eqs. (150 and 151), we have

$$\frac{3M_{J/\psi}(1S) + M_{\eta_c}(1S)}{4} = \frac{3}{2} \lambda_{22} + C' \quad (152)$$

and

$$\frac{3M_{\psi}(2S) + M_{\eta_c}(2S)}{4} = \frac{7}{2} \lambda_{22} + C' \quad (153)$$

Where,  $C' = -\frac{4}{3} \bar{C} + 2m_c$ . The factor 7/2 with  $\lambda_{22}$  in above equation comes from the relation  $E_{nlm} = \lambda_{22}(4n + 2l + 3)/2$  [49] for  $n = 1$  and  $l = 0$ . Whereas in eq. (152) above, the factor 3/2 with  $\lambda_{22}$  is for  $n = 0$  and  $l = 0$ .

Put the values of masses  $M_{J/\psi}(1S) = 3.096916 \text{ GeV}$ ,  $M_{\eta_c}(1S) = 2.9803 \text{ GeV}$ ,  $M_{\psi}(2S) = 3.68609 \text{ GeV}$  and  $M_{\eta_c}(2S) = 3.637 \text{ GeV}$  from (PDG) ref. [9] in eqs. (152) and (153) and solving them simultaneously, we get  $\lambda_{22} = 0.303028 \text{ GeV}$  and  $C' = 2.61322 \text{ GeV}$ . We took constituent mass of light quark as  $m = 0.33 \text{ GeV}$  and that of charm quark  $m_c = 1.4794 \text{ GeV}$  [50]. This gives  $\bar{C} = 0.259185 \text{ GeV}$ . The relation  $\lambda_{22} = 1/m_c d_{22}^2$  then gives the value of  $d_{22}$  or  $d'' = 0.294583 \text{ fm}$  and using this value in eq. (10) gives the value of  $d_{21} = d = 0.428648 \text{ fm}$  and by putting this value of  $d$  in eq. (9) we get  $d_{11} = d_{12} = d' = 0.379058 \text{ fm}$ . By putting the above values of  $d_{22}$  and  $\lambda_{22}$  in eq. (151) we get the value of the parameter  $C = -0.0254714 \text{ GeV}^3$ . The value of  $b_s$  is taken as  $0.18 \text{ GeV}^2$  [47]. We take the value of  $k_f = 0.075$  [48].

## 4.2 Finding the Phase Shifts, Cross Sections and Binding

To find out the phase shifts we have used the same relation

$$s = I - 2iT \quad (154)$$

and [51]

$$s = \exp(2i\Delta) \quad (155)$$

and by comparing their right hand sides, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2i \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2i \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \quad (156)$$

$\delta_{ij} = -T_{ij}$  for  $i, j = 1, 2$  [51]. Because the phase shifts are much less than a radian, so we can neglect the higher powers of exponential series for eq. (155) above. To find out the total cross sections we have used eqs. [27 to 29] of ref. [52] while according to our definition of Fourier transform  $h_{fi}$  has been taken equal to  $T_{fi}$  [53] [54] for  $i, f = 1, 2$ . The total cross-section is  $4\pi$  times differential cross section for any value of centre of mass kinetic energy,  $T_c$ . The condition for the existence of a bound state of the  $q^2\bar{q}^2$  system is [39]

$$\det W = 0. \quad (157)$$

For the definition of matrix  $W$ , see eq. (134).

As this is a determinant of a  $4 \times 4$  matrix, once we select the numerical values of the parameters in it, it would have more than one values of the energy that satisfies it.

## 5 Results and Conclusion

Fig. 3 represents the elastic scattering phase shifts in degrees against centre of mass kinetic energies in  $GeV$  for the process  $D^o\bar{D}^{o*} \rightarrow D^o\bar{D}^{o*}$ , corresponding to t-matrix element given in eq. (145). Fig. 5 represents the elastic scattering phase shifts for the process  $\omega J/\psi \rightarrow \omega J/\psi$ , corresponding to t-matrix element given in eq. (148). In fig. 4 we have shown the inelastic scattering phase shifts for the process  $D^o\bar{D}^{o*} \rightarrow \omega J/\psi$ , corresponding to t-matrix element given in eq. (146). The same results are obtained for the inelastic scattering process  $\omega J/\psi \rightarrow D^o\bar{D}^{o*}$ , corresponding to t-matrix element given in eq. (149). The reason for selecting  $\omega$  meson in the scattering process is that the mass sum of  $\omega$  and  $J/\psi$  is almost equal to the mass of  $X(3872)$  meson. To find out these phase shifts we have used the relations, between s-matrix elements and corresponding phase shifts, given in the reference [51].

We have found a bound state (see eq. (157), condition for the existence of a bound state) at total centre of mass energy  $E_c = 3.6942 GeV$ . But we can get a bound state at  $E_c = 3.872 GeV$  by multiplying the off-diagonal terms of potential energy, kinetic energy and normalization matrices (i.e. multiplying  $l_o$  of eqs. (92 and 93)) by a factor- $Y_o$  for value 1.97. See fig. 6 where we have plotted the factor  $Y_o$  against total centre of mass energy for a possible bound state. (As the equation, last in the above section, that gives this energy value can have more than ne solutions there may be other curves for factors like this  $Y_o$  as well.) This bound state at  $E_c = 3.872 GeV$  may be recognized as  $X(3872)$ . In fig. 4 we have -ve phase shifts indicating an attractive potential so the possibility of  $D^o\bar{D}^{o*}$  bound state being  $X(3872)$  is very much alive. Alternatively if we take the value of  $k_f$  being 0.0048 instead of 0.075 then we can get a bound state even at  $3.872 GeV$ . So by adjusting the free parameter  $k_f$  we can get the bound state at  $3.872 GeV$ .

In figs. (7, 8 and 9) we have shown the total cross sections for the scattering of  $D^o\bar{D}^{o*} \rightarrow D^o\bar{D}^{o*}$ ,  $D^o\bar{D}^{o*} \rightarrow \omega J/\psi$  and  $\omega J/\psi \rightarrow \omega J/\psi$ , if the same eqs. [27 to 29] of ref. [52] are used for the inelastic process as well. These results can be compared with the experiments. We have also calculated the total cross sections for  $D^o\bar{D}^{o*} \rightarrow J/\psi\pi^o$  and vice versa scattering.

The same technique may be used to study the other scalar mesons e.g.  $X(3940)$ ,  $Y(4260)$  and  $Z(4433)$ . In future we plan to study the structure of scalar mesons by using the more realistic linear-plus-columbic

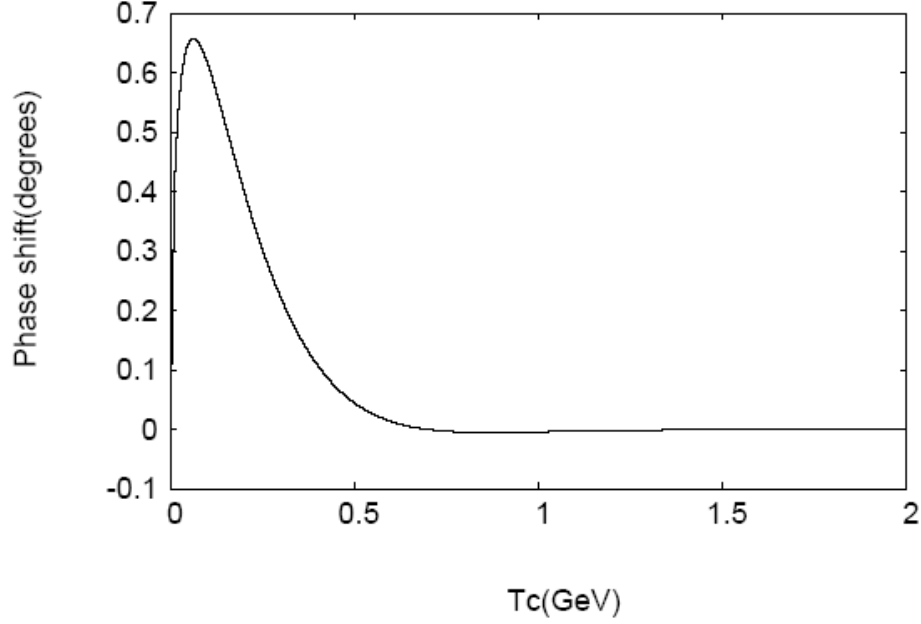


Figure 3: Elastic scattering phase shifts for the process  $D^0 \bar{D}_o^* \rightarrow D^0 \bar{D}_o^*$ .

potential and corresponding wave functions along with the use of the surface area model of  $f$  of eq.23 and not Gaussian  $f$  of eq. 24. In figs. (11 to 13) phase shifts are shown for  $k_f = 0$ , while in figs. (14 to 16) total cross section are given for  $k_f = 0$ . It is to be noted that both phase shifts and the cross sections decrease when we use the QCD effects in the form of  $f$ .

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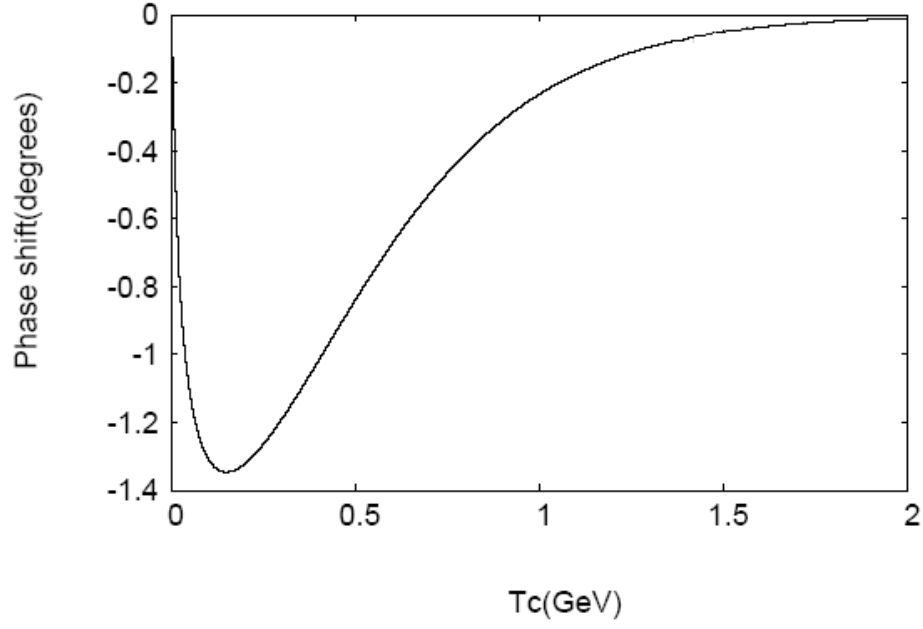


Figure 4: Inelastic scattering phase shifts for the process  $D^o\overline{D}^{o*} \longrightarrow \omega J/\psi$ . We checked that the same results are obtained for the inelastic scattering process  $\omega J/\psi \longrightarrow D^o\overline{D}^{o*}$ .

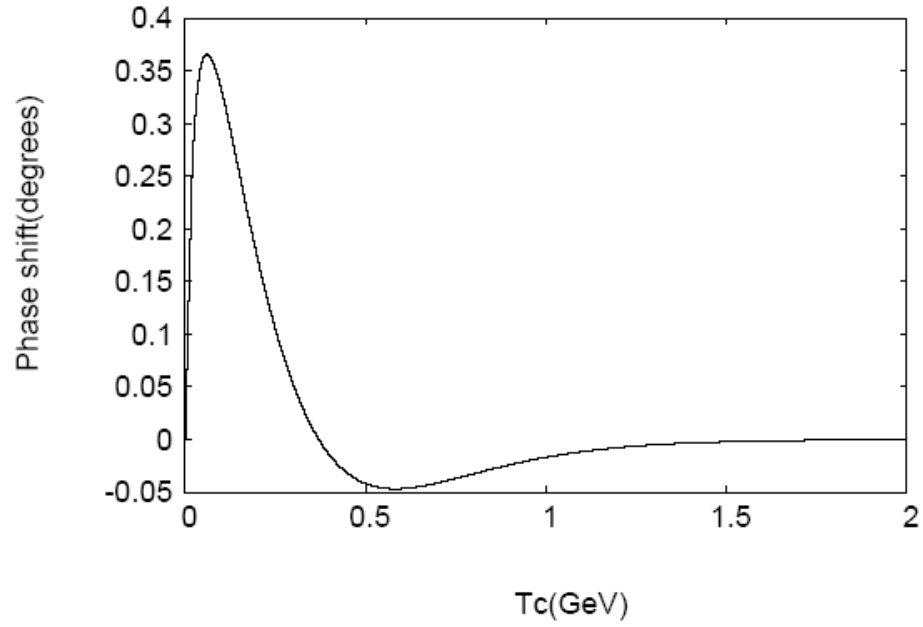


Figure 5: Elastic scattering phase shifts for the process  $\omega J/\psi \longrightarrow \omega J/\psi$

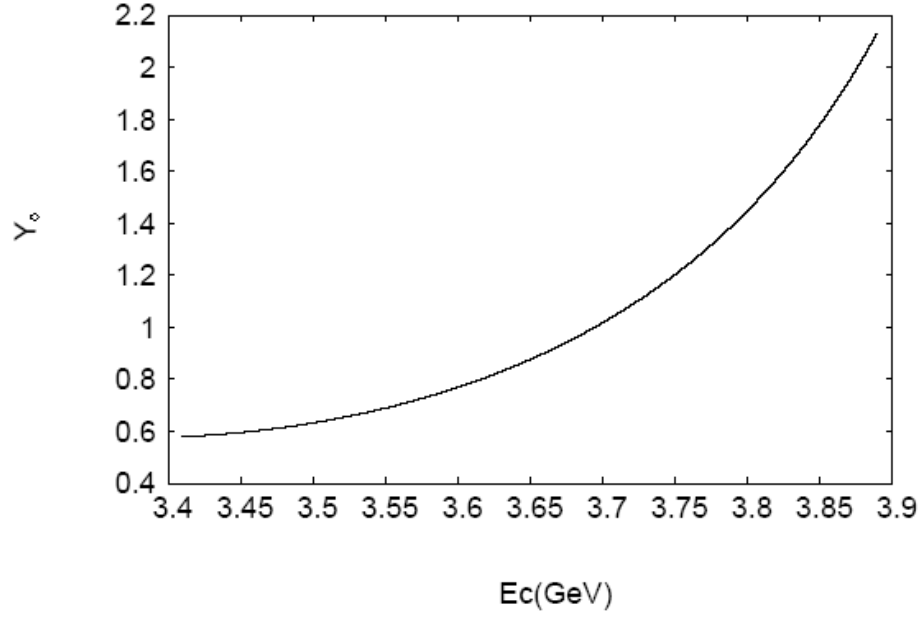


Figure 6: Factor  $Y_o$  to get binding, versus  $E_c$ . As mentioned in the text, there may be other values of it for the same energy.

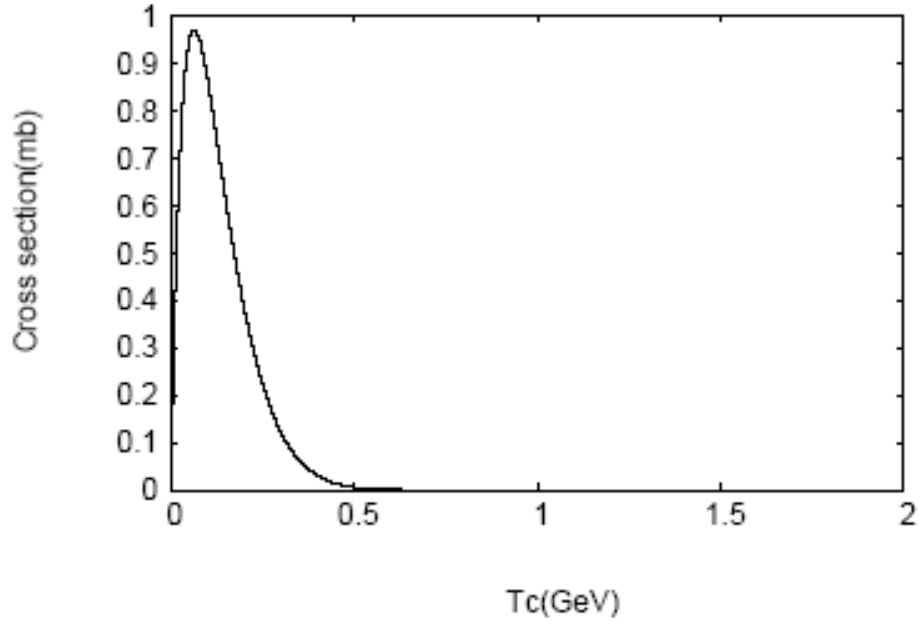


Figure 7: Total cross sections for the scattering process  $D^o \overline{D}_o^* \rightarrow D^o \overline{D}_o^*$ .

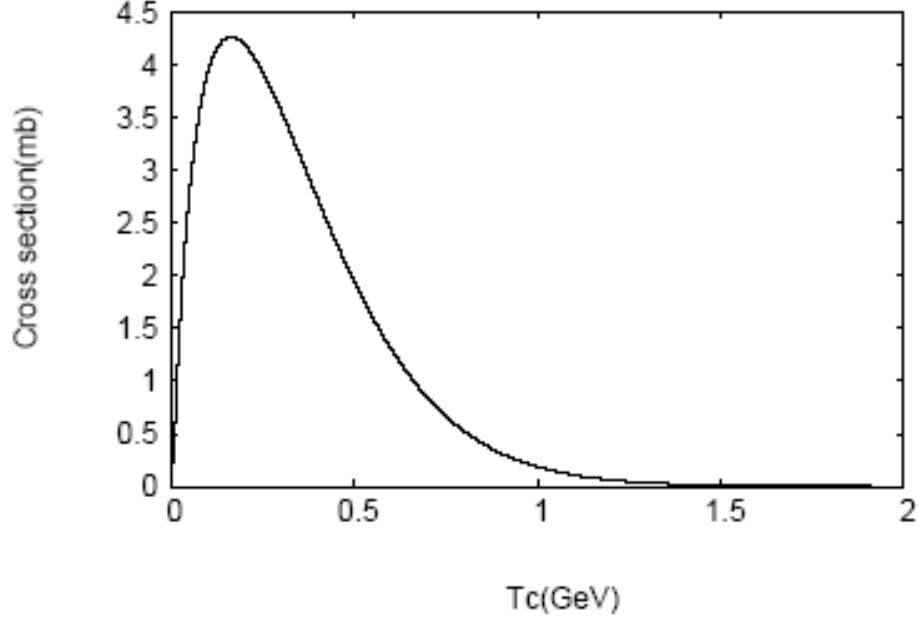


Figure 8: Total cross sections for the scattering process  $D^0 \bar{D}^{0*} \rightarrow \omega J/\psi$ , if the same eqs. [27 to 29] of ref. [52] are used for the inelastic process as well. We checked that the same results are obtained for the inelastic scattering process  $\omega J/\psi \rightarrow D^0 \bar{D}^{0*}$ .

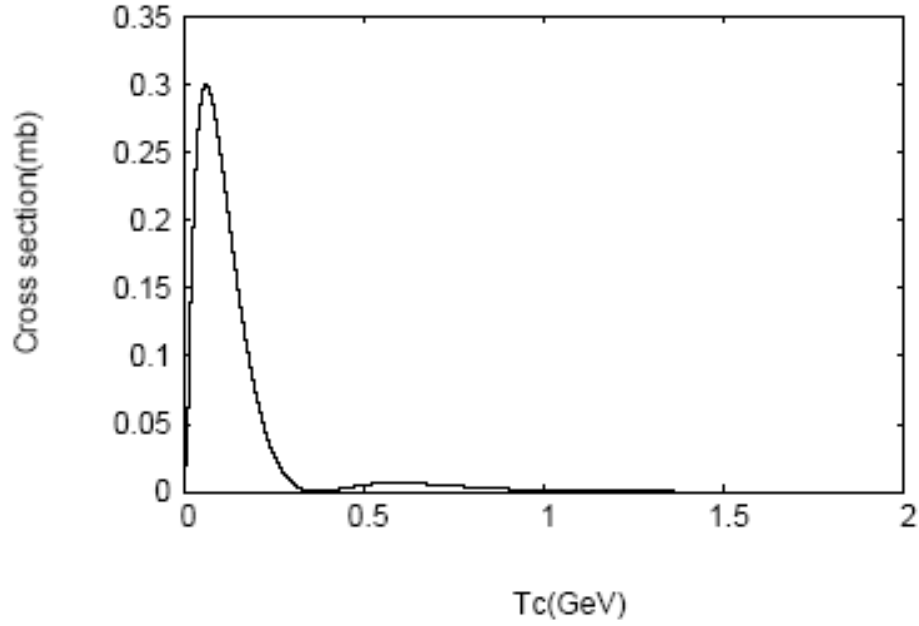


Figure 9: Total cross sections for the process  $\omega J/\psi \rightarrow \omega J/\psi$

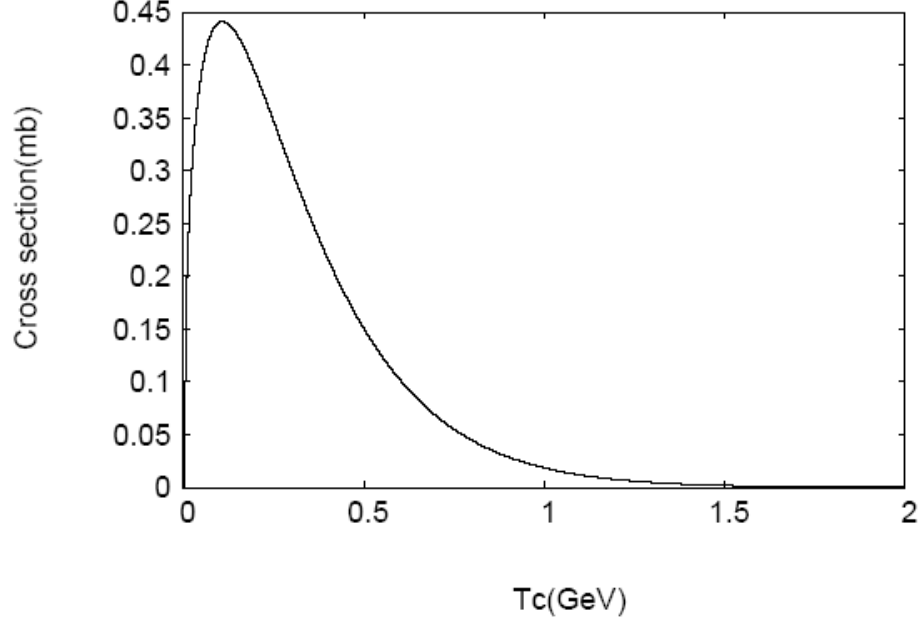


Figure 10: Total cross sections for the scattering process  $D^0\overline{D}^{*0} \rightarrow J/\psi\pi$ , if the same eqs. [27 to 29] of ref. [52] are used for the inelastic process as well. The same results are obtained for the scattering process  $J/\psi\pi \rightarrow D^0\overline{D}^{*0}$ .

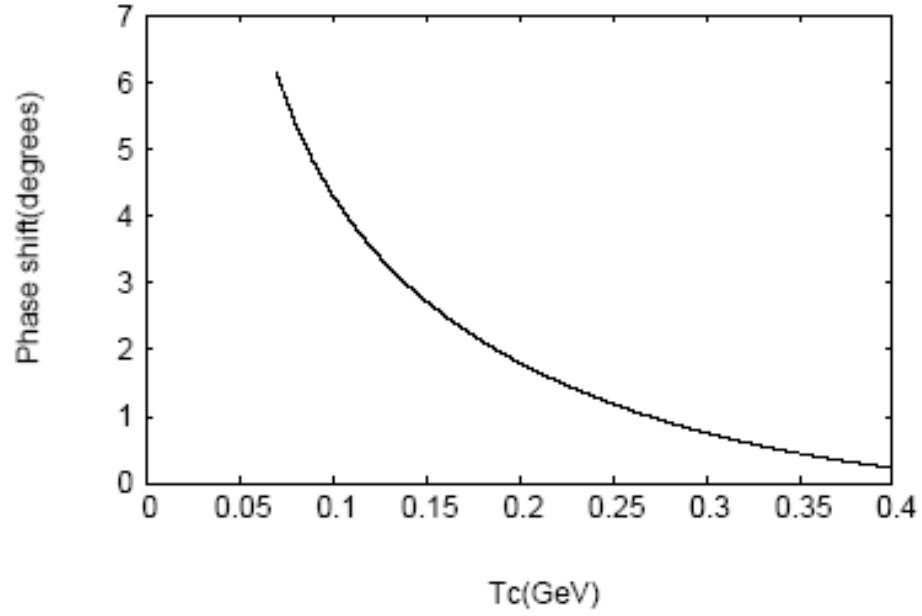


Figure 11: Elastic scattering phase shifts for the process  $D^0\overline{D}^{*0} \rightarrow D^0\overline{D}_o^{*0}$ , for  $k_f = 0$ .



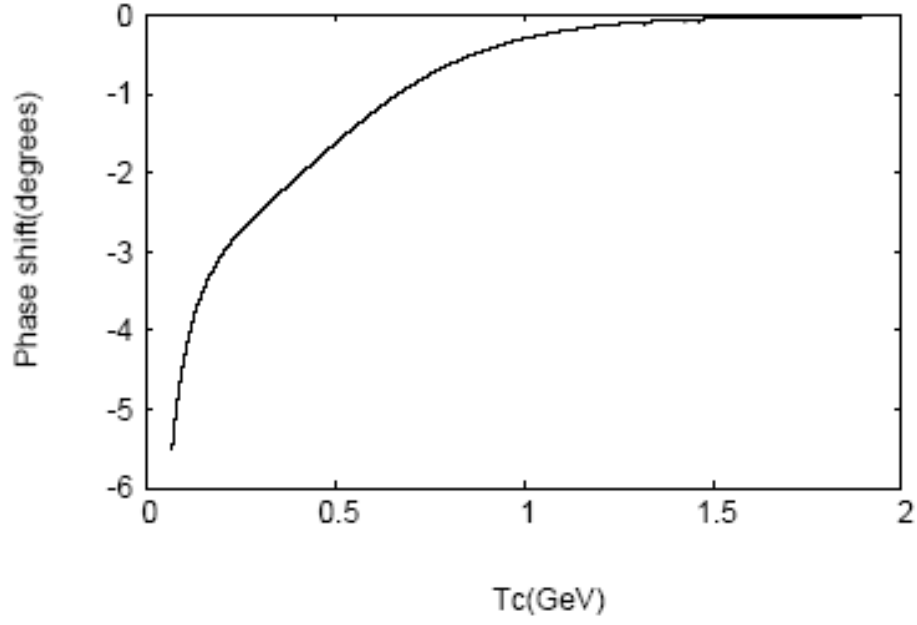


Figure 12: Inelastic scattering phase shifts for the process  $D^o\overline{D}^{o*} \rightarrow \omega J/\psi$ . We checked that the same results are obtained for the inelastic scattering process  $\omega J/\psi \rightarrow D^o\overline{D}^{o*}$ , for  $k_f = 0$ .

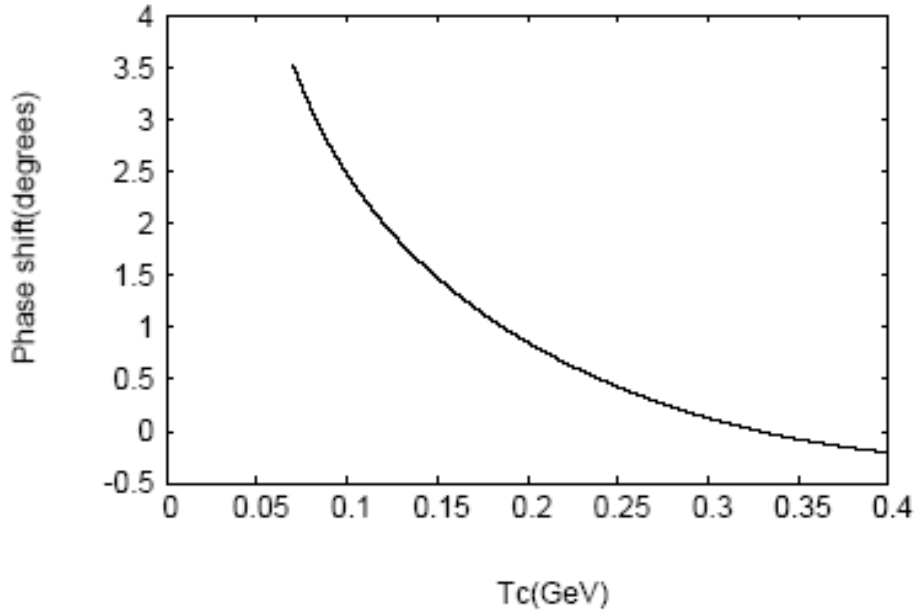


Figure 13: Elastic scattering phase shifts for the process  $\omega J/\psi \rightarrow \omega J/\psi$ , for  $k_f = 0$ .

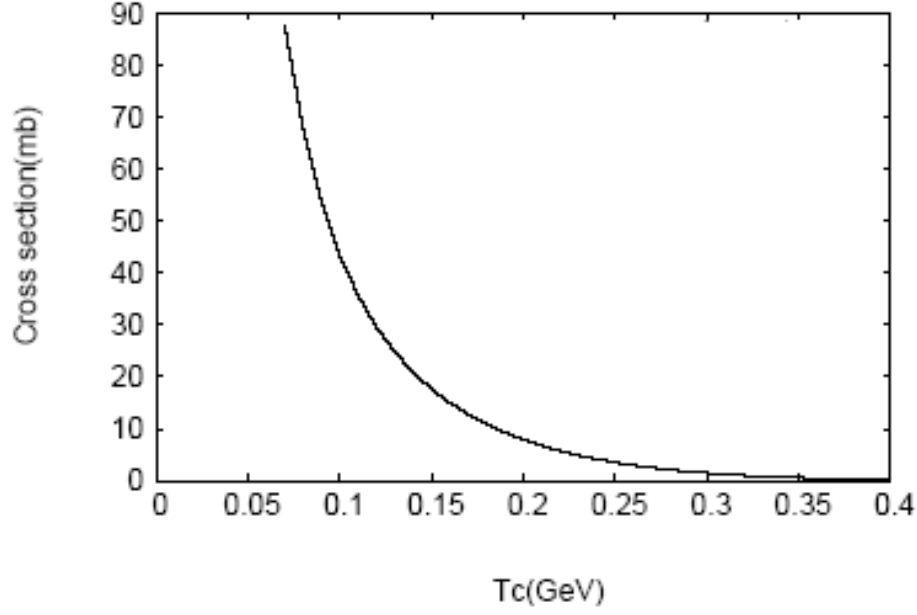


Figure 14: Total cross sections for the scattering process  $D^0 \bar{D}_o^* \longrightarrow D^0 \bar{D}_o^*$ , for  $k_f = 0$ .

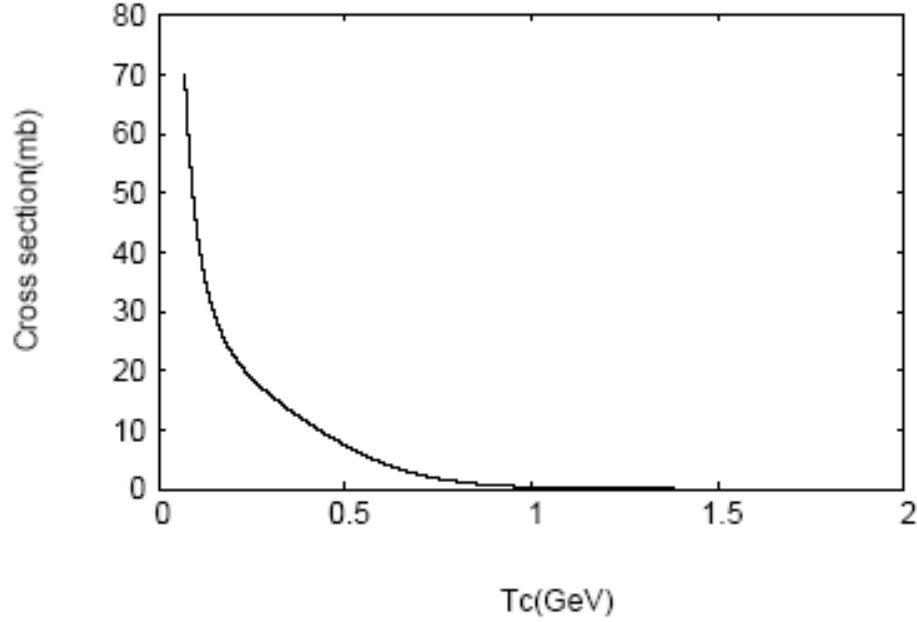


Figure 15: Total cross sections for the scattering process  $D^0 \bar{D}^{o*} \longrightarrow \omega J/\psi$ , if the same eqs. [27 to 29] of ref. [52] are used for the inelastic process as well. We checked that the same results are obtained for the inelastic scattering process  $\omega J/\psi \longrightarrow D^0 \bar{D}^{o*}$ , for  $k_f = 0$ .

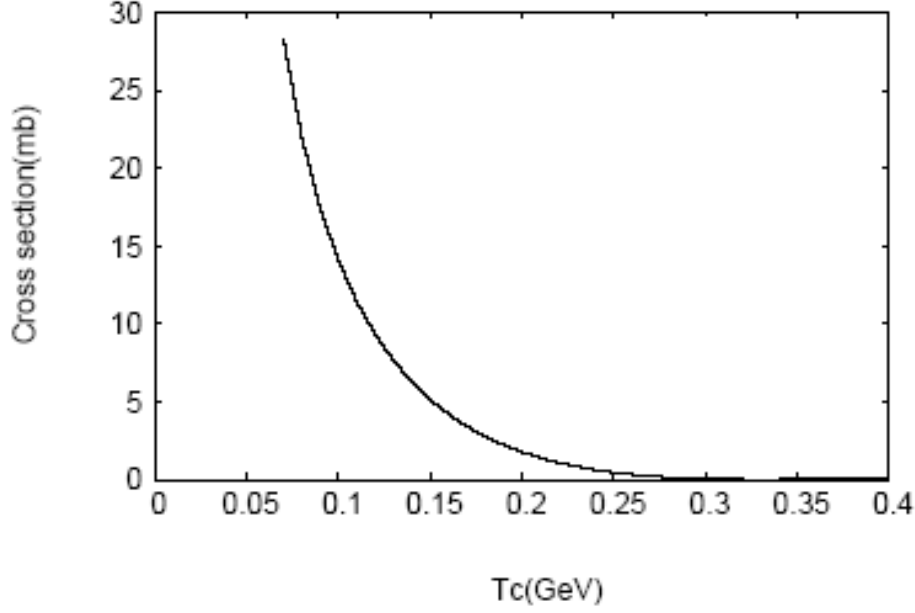


Figure 16: Total cross sections for the process  $\omega J/\psi \rightarrow \omega J/\psi$ , for  $k_f = 0$ .

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## Appendix A: The spin basis

For total spin 1 different possible spin states for a meson meson molecule are

$$|V_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{3}}(|1, 1\rangle + |1, 0\rangle + |1, -1\rangle),$$

$V$  stands for a vector meson. Inserting the completeness expression for an identity  $I$  the above equation may be written as

$$\begin{aligned} |V_{1\bar{3}}V_{2\bar{4}}\rangle &= \frac{1}{\sqrt{3}}(|1, 1, 1, 0\rangle\langle 1, 1, 1, 0| + |1, 0, 1, 1\rangle\langle 1, 0, 1, 1|)|1, 1\rangle + \frac{1}{\sqrt{3}}(|1, -1, 1, 1\rangle \\ &\quad \langle 1, -1, 1, 1| + |1, 1, 1, -1\rangle\langle 1, 1, 1, -1| + |1, 0, 1, 0\rangle\langle 1, 0, 1, 0|)|1, 0\rangle \\ &\quad + \frac{1}{\sqrt{3}}(|1, 0, 1, -1\rangle\langle 1, 0, 1, -1| + |1, -1, 1, 0\rangle\langle 1, -1, 1, 0|)|1, -1\rangle. \end{aligned}$$

Using Clebsch-Gordan coefficients it becomes

$$\begin{aligned} |V_{1\bar{3}}V_{2\bar{4}}\rangle &= \frac{1}{\sqrt{6}}(|1, 1\rangle|1, 0\rangle - |1, 0\rangle|1, 1\rangle + |1, 1\rangle|1, -1\rangle - |1, -1\rangle|1, 1\rangle + \\ &\quad |1, 0\rangle|1, -1\rangle - |1, -1\rangle|1, 0\rangle). \end{aligned}$$

Again using identity

$$I = \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right| + \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right| + \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right| + \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right|,$$

and using the Clebsch-Gordan coefficients we get

$$|V_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{12}} \left[ \uparrow\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\uparrow + \sqrt{2} \uparrow\downarrow\uparrow\downarrow - \sqrt{2} \downarrow\uparrow\downarrow\uparrow + \uparrow\downarrow\downarrow\downarrow + \downarrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\downarrow - \downarrow\downarrow\downarrow\uparrow \right]. \quad (158)$$

From eq. (158) by interchanging positions of  $\bar{3}$  and  $\bar{4}$ , we have

$$|V_{1\bar{4}}V_{2\bar{3}}\rangle = \frac{1}{\sqrt{12}} \left[ \uparrow\uparrow\downarrow\uparrow + \uparrow\downarrow\uparrow\uparrow - \uparrow\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\uparrow + \sqrt{2} \uparrow\downarrow\downarrow\uparrow - \sqrt{2} \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\downarrow + \downarrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\downarrow \right]. \quad (159)$$

Now again for total spin 1, we have

$$|P_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{3}} \left( |1, 1\rangle + |1, 0\rangle + |1, -1\rangle \right),$$

where  $P$  stands for a pseudoscalar meson. Inserting the completeness expression for an identity  $I$  the above equation may be written as

$$|P_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{3}} \left( |0, 0, 1, 1\rangle \langle 0, 0, 1, 1| \right) |1, 1\rangle + \frac{1}{\sqrt{3}} \left( |0, 0, 1, 0\rangle \langle 0, 0, 1, 0| \right) |1, 0\rangle + \frac{1}{\sqrt{3}} \left( |0, 0, 1, -1\rangle \langle 0, 0, 1, -1| \right) |1, -1\rangle.$$

Using the Clebsch-Gordan coefficients it becomes

$$|P_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{3}} \left( |0, 0\rangle |1, 1\rangle + |0, 0\rangle |1, 0\rangle + |0, 0\rangle |1, -1\rangle \right).$$

Using the identity

$$I = \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right| + \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right| + \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right| + \left| \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right|,$$

and using the Clebsch-Gordan coefficients we get

$$|P_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{6}} \left[ \uparrow\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\uparrow + \frac{1}{\sqrt{2}} \left( \uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow - \downarrow\downarrow\uparrow\uparrow \right) + \uparrow\downarrow\downarrow\downarrow - \downarrow\downarrow\uparrow\downarrow \right] \quad (160)$$

In the same way by inserting the completeness expressions for identity  $I$  and using the corresponding Clebsch-Gordan coefficients we can have

$$|V_{1\bar{3}}P_{2\bar{4}}\rangle = \frac{1}{\sqrt{6}} \left[ \uparrow\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow\uparrow + \frac{1}{\sqrt{2}} \left( \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\uparrow\downarrow - \downarrow\downarrow\uparrow\uparrow \right) + \uparrow\downarrow\downarrow\downarrow - \downarrow\downarrow\uparrow\downarrow \right] \quad (161)$$

$$|P_{1\bar{4}}V_{2\bar{3}}\rangle = \frac{1}{\sqrt{6}} \left[ \uparrow\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\uparrow + \frac{1}{\sqrt{2}} \left( \uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow \right) + \uparrow\downarrow\downarrow\downarrow - \downarrow\downarrow\downarrow\uparrow \right] \quad (162)$$

$$|V_{1\bar{4}}P_{2\bar{3}}\rangle = \frac{1}{\sqrt{6}} \left[ \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow + \frac{1}{\sqrt{2}} \left( \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow \right) + \downarrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\downarrow \right]. \quad (163)$$

Similarly for total spin 0 we have the following possible spin states for a meson meson molecule

$$|V_{1\bar{3}}V_{2\bar{4}}\rangle = \frac{1}{\sqrt{12}} \left[ 2 \uparrow\downarrow\uparrow\downarrow + 2 \downarrow\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow - \downarrow\downarrow\uparrow\uparrow \right] \quad (164)$$

$$|V_{1\bar{4}}V_{2\bar{3}}\rangle = \frac{1}{\sqrt{12}} \left[ 2 \uparrow\downarrow\downarrow\uparrow + 2 \downarrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow \right] \quad (165)$$

$$|P_{1\bar{3}}P_{2\bar{4}}\rangle = \frac{1}{2} \left[ \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow \right] \quad (166)$$

$$|P_{1\bar{4}}P_{2\bar{3}}\rangle = \frac{1}{2} \left[ \uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow \right]. \quad (167)$$

Overlap of these spin states from eq. (158) to eq. (167) are shown in table 1 and table 2.

| For Spin 1                 |                            |                            |                            |                            |                            |                            |
|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
|                            | $V_{1\bar{3}}V_{2\bar{4}}$ | $V_{1\bar{4}}V_{2\bar{3}}$ | $P_{1\bar{3}}V_{2\bar{4}}$ | $V_{1\bar{3}}P_{2\bar{4}}$ | $P_{1\bar{4}}V_{2\bar{3}}$ | $V_{1\bar{4}}P_{2\bar{3}}$ |
| $V_{1\bar{3}}V_{2\bar{4}}$ | 1                          | 0                          | 0                          | 0                          | $\frac{1}{\sqrt{2}}$       | $-\frac{1}{\sqrt{2}}$      |
| $V_{1\bar{4}}V_{2\bar{3}}$ | 0                          | 1                          | $\frac{1}{\sqrt{2}}$       | $-\frac{1}{\sqrt{2}}$      | 0                          | 0                          |
| $P_{1\bar{3}}V_{2\bar{4}}$ | 0                          | $\frac{1}{\sqrt{2}}$       | 1                          | 0                          | $\frac{1}{2}$              | $\frac{1}{2}$              |
| $V_{1\bar{3}}P_{2\bar{4}}$ | 0                          | $-\frac{1}{\sqrt{2}}$      | 0                          | 1                          | $\frac{1}{2}$              | $\frac{1}{2}$              |
| $P_{1\bar{4}}V_{2\bar{3}}$ | $\frac{1}{\sqrt{2}}$       | 0                          | $\frac{1}{2}$              | $\frac{1}{2}$              | 1                          | 0                          |
| $V_{1\bar{4}}P_{2\bar{3}}$ | $-\frac{1}{\sqrt{2}}$      | 0                          | $\frac{1}{2}$              | $\frac{1}{2}$              | 0                          | 1                          |

Table 1: Spin overlaps for spin-1, Meson-meson systems. Here  $P$  stands for pseudoscalar and  $V$  for vector mesons.

| For Spin 0                 |                            |                            |                            |                            |
|----------------------------|----------------------------|----------------------------|----------------------------|----------------------------|
|                            | $V_{1\bar{3}}V_{2\bar{4}}$ | $V_{1\bar{4}}V_{2\bar{3}}$ | $P_{1\bar{3}}P_{2\bar{4}}$ | $P_{1\bar{4}}P_{2\bar{3}}$ |
| $V_{1\bar{3}}V_{2\bar{4}}$ | 1                          | $-\frac{1}{2}$             | 0                          | $-\frac{\sqrt{3}}{2}$      |
| $V_{1\bar{4}}V_{2\bar{3}}$ | $-\frac{1}{2}$             | 1                          | $-\frac{\sqrt{3}}{2}$      | 0                          |
| $P_{1\bar{3}}P_{2\bar{4}}$ | 0                          | $-\frac{\sqrt{3}}{2}$      | 1                          | $\frac{1}{2}$              |
| $P_{1\bar{4}}P_{2\bar{3}}$ | $-\frac{\sqrt{3}}{2}$      | 0                          | $\frac{1}{2}$              | 1                          |

Table 2: Spin overlaps for spin-0, Meson-meson systems. Here  $P$  stands for pseudoscalar and  $V$  for vector mesons (These agree with eqs. D5 to D8 of appendix D of ref. [26]).

## Appendix B: The colour basis

The two meson-like base states  $\{|1_{1\bar{3}}1_{2\bar{4}}\rangle_c, |1_{1\bar{4}}1_{2\bar{3}}\rangle_c\}$  in terms of orthonormal basis  $\{|\bar{3}_{12}3_{\bar{3}\bar{4}}\rangle_c, |6_{12}\bar{6}_{\bar{3}\bar{4}}\rangle_c\}$  are

$$|1_{1\bar{3}}1_{2\bar{4}}\rangle_c = \frac{1}{\sqrt{3}}|\bar{3}_{12}3_{\bar{3}\bar{4}}\rangle_c + \sqrt{\frac{2}{3}}|6_{12}\bar{6}_{\bar{3}\bar{4}}\rangle_c \quad (168)$$

$$|1_{1\bar{4}}1_{2\bar{3}}\rangle_c = -\frac{1}{\sqrt{3}}|\bar{3}_{12}3_{\bar{3}\bar{4}}\rangle_c + \sqrt{\frac{2}{3}}|6_{12}\bar{6}_{\bar{3}\bar{4}}\rangle_c \quad (169)$$

The matrix elements of the identity operator in the new basis  $\{|1_{1\bar{3}}1_{2\bar{4}}\rangle_c, |1_{1\bar{4}}1_{2\bar{3}}\rangle_c\}$  are shown in eq. (20), taking  $f = 1$ . The matrix elements of colour Casimir operator  $\mathbf{F}_i \cdot \mathbf{F}_j$  in the new colour basis  $\{|1_{1\bar{3}}1_{2\bar{4}}\rangle_c, |1_{1\bar{4}}1_{2\bar{3}}\rangle_c\}$  are (see C13 to C15 of appendix-C of ref. [26])

$$\begin{aligned} {}_c\langle 1 | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 1 \rangle_c &= {}_c\langle 1_{1\bar{3}}1_{2\bar{4}} | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 1_{1\bar{3}}1_{2\bar{4}} \rangle_c = \begin{pmatrix} 0 \\ -\frac{4}{3} \\ 0 \\ 0 \\ -\frac{4}{3} \\ 0 \end{pmatrix} \\ {}_c\langle 2 | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 2 \rangle_c &= {}_c\langle 1_{1\bar{4}}1_{2\bar{3}} | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 1_{1\bar{4}}1_{2\bar{3}} \rangle_c = \begin{pmatrix} 0 \\ 0 \\ -\frac{4}{3} \\ -\frac{4}{3} \\ 0 \\ 0 \end{pmatrix} \\ {}_c\langle 1 | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 2 \rangle_c &= {}_c\langle 2 | \begin{pmatrix} \mathbf{F}_1 \cdot \mathbf{F}_2 \\ \mathbf{F}_1 \cdot \mathbf{F}_3 \\ \mathbf{F}_1 \cdot \mathbf{F}_4 \\ \mathbf{F}_2 \cdot \mathbf{F}_3 \\ \mathbf{F}_2 \cdot \mathbf{F}_4 \\ \mathbf{F}_3 \cdot \mathbf{F}_4 \end{pmatrix} | 1 \rangle_c = \begin{pmatrix} \frac{4}{9} \\ -\frac{4}{9} \\ -\frac{4}{9} \\ -\frac{4}{9} \\ -\frac{4}{9} \\ \frac{4}{9} \end{pmatrix} \end{aligned}$$